Symbolic and Graphical Investigations of Riemann Sums with a Computer Algebra System

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ABSTRACT: This article explores the ways that a computer algebra system may be used in conjunction with plotting software to allow students to investigate the convergence properties of Riemann sums for functions of one or two variables in a way that is not possible with just numerical calculations. Explicit formulas for the Riemann sums suggest algebraic combinations that lead to the trapezoid and Simpson's rules as a way to improve the rate of convergence.

KEYWORDS: Riemann sums, trapezoid rule, Simpson's rule, rates of convergence.

In many recent calculus texts (and some of the older ones, too!), the definite integral is first thought of geometrically as area, and some form of the Fundamental Theorem is often obtained through constructing and solving a differential equation. Only later is a connection made with the analytic definition of the integral as the limit of a Riemann sum. One of the main stumbling blocks with using Riemann sums to define the definite integral for beginning calculus students has been the difficulty in expressing the Riemann sum in a closed form that permits the evaluation of the limit. Without this step we have been forced to rely mainly on numerical calculations and animated pictures of rectangles "filling" the region under the curve as more and more (thinner) rectangles are drawn to convince our students that these sums actually do converge. Computer algebra systems such as Derive, Mathematica, and Maple, however, can easily simplify Riemann sums for all polynomials and for functions of the form $\sin(mx)$, $\cos(mx)$, and e^{mx}, thereby providing a wealth of examples involving the standard functions of calculus that students can use to investigate the convergence properties of these sums. In particular, we would like our students to

- understand the definition of a Riemann sum;
- demonstrate that the limit of a Riemann sum is the same for (almost) any choice of points from each subinterval;
- appreciate the different rates of convergence of the approximating sums; and
- investigate combinations of these Riemann sums that lead to better approximations.

The following examples illustrate some of the investigations I use with students to achieve these goals.

Example 1. Polynomials $\int_0^2 \left(x^4 + 2x^3 + x^2 - 3x + 1\right) dx$

Our calculus classes meet in a computerized classroom to work on special projects and assignments. I first have the students investigate a common function such as this one. We construct the Riemann sum for this integral using n subintervals of uniform width Δx over the interval [0, 2]. I purposely chose the interval of integration to start at 0 to simplify the sum, but the same ideas can be explored using any interval. With $F(x) = x^4 + 2x^3 + x^2 - 3x + 1$, we use our computer algebra system[1] to compute the Riemann sum using the right endpoint of each subinterval and obtain

$$\sum_{k=1}^{n} F(k\Delta x)\,\Delta x = \Delta x\, n\,[\Delta x^4(n+1)(6n^3 + 9n^2 + n - 1) + 15\Delta x^3 n(n+1)^2 + 5\Delta x^2(n+1)(2n+1) - 45\Delta x(n+1) + 30] \big/ 30\,.$$

At this point the expression is still written in terms of both Δx and n. A choice needs to be made to either substitute for Δx in terms of n, or to substitute for n in terms of Δx. The former would led to the limit as n approaches infinity, while the latter would be used for the limit as Δx approaches 0. Simplifying the expression obtained by substituting $n = 2/\Delta x$ gives the right Riemann sum as

$$\text{Right}(\Delta x) = \sum_{k=1}^{n} F(k\Delta x)\,\Delta x = \frac{196}{15} + 15\Delta x + 5\Delta x^2 - \frac{\Delta x^4}{15}.$$

Repeating the same steps for the Riemann sum using the left endpoint of each of the n subintervals gives

$$\text{Left}(\Delta x) = \sum_{k=1}^{n} F((k-1)\Delta x)\,\Delta x = \frac{196}{15} - 15\Delta x + 5\Delta x^2 - \frac{\Delta x^4}{15}.$$

In both cases we easily see that the limit of the Riemann sum as Δx goes to 0 is $196/15$. Moreover, the two expressions differ only in the Δx term! This, of course, is not surprising once the students realize that the only difference between the right Riemann sum and the left Riemann sum is at the two endpoints of the interval of integration, so that $\text{Right}(\Delta x) - \text{Left}(\Delta x) = \left(F(2) - F(0)\right)\Delta x$. Another interesting observation—there is no Δx^3 term. A coincidence?

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[1]We use Derive in the classroom but the results shown in this paper were obtained using Mathematica 2.2 for the Macintosh. To compute the sum on the interval `[a,b]` in Derive, first declare `dx` to be a positive constant, author the function `F`, then author `sum(F(a+k*dx)dx,k,1,n)`. Simplify the result, substitute (b-a)/dx for n, then expand this result. In Mathematica, load the SymbolicSum package, define the function `F`, then compute `Apart[Simplify[Sum[F[a+k*dx]*dx,{k,1,n}]/.{n->(b-a)/dx}]]`.

While these expressions for the Riemann sums are only valid for Δx of the form $2/n$ for an integer value of n, it is nevertheless interesting to view them as functions of Δx and to draw their graphs on the interval [0, 2]. The upper curve in Figure 1 is the graph of Right(Δx) and the lower curve is that of Left(Δx). After I have the students generate these curves, we discuss how the limits found from the symbolic sums are reflected in the graph. What is clear from these graphs is that both sums converge to the same limit but at an apparent slow rate as evidenced by the slope of each curve as Δx approaches 0. To make this idea more precise, we discuss how the difference between each Riemann sum and the limiting value 196/15, that is, the error in the approximation, is controlled primarily by the linear terms $\pm 15\Delta x$ for values of Δx near 0.

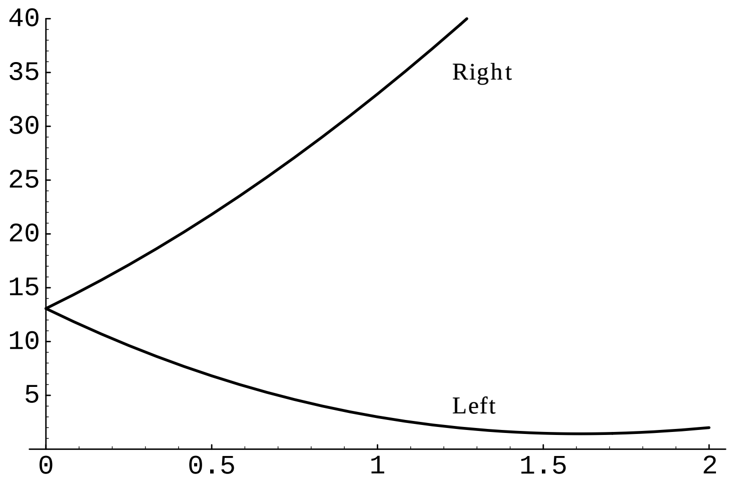

FIGURE 1
The right and left Riemann sums as functions of Δx.

A Riemann sum is supposed to converge to the same limit for any choice of points from each subinterval. While it is not possible to have a computer algebra system compute a closed-form expression for a sum using randomly chosen points from each subinterval, we *can* ask for the sum if we use a consistent method for choosing the points. In particular, if we let the points be chosen at a distance of $s\Delta x$ from the right endpoint of each subinterval where s ranges between 0 (corresponding to the right endpoints) and 1 (indicating left endpoints), then we get an “arbitrary” Riemann sum

$$\sum_{k=1}^{n} F((k-s)\Delta x)\,\Delta x = \frac{196}{15} + (15-30s)\Delta x + (5-30s+30s^2)\Delta x^2 + (-6s+18s^2-12s^3)\Delta x^3$$
$$+ (-1/15 + 2s^2 - 4s^3 + 2s^4)\Delta x^4\,.$$

The students now observe that it is indeed the case that this sum converges to 196/15 for every choice of s as Δx goes to 0. Another observation is that for $s = 1/2$ the coefficient of Δx vanishes and the error in the approximation is now controlled primarily by a quadratic term in Δx. This suggests that the midpoint Riemann sum

$$\mathrm{Mid}(\Delta x) = \sum_{k=1}^{n} F((k - \tfrac{1}{2})\Delta x)\, \Delta x = \frac{196}{15} - \frac{5\Delta x^2}{2} + \frac{7\Delta x^4}{120}$$

should provide a better approximation to the value of the integral than using the right or left endpoints. (As a side bonus, the Δx^3 term has also dropped out.) Finally, a closer look at the expressions for the right and left Riemann sums reveals that the linear term may also be eliminated by taking the average of the two sums, yielding the trapezoid rule

$$\mathrm{Trap}(\Delta x) = \frac{\mathrm{Right}(\Delta x) + \mathrm{Left}(\Delta x)}{2} = \frac{196}{15} + 5\Delta x^2 - \frac{\Delta x^4}{15}.$$

In Figure 2, I have the students compare the graphs of the three Riemann sums and the trapezoid rule as functions of Δx. Figures 3 and 4 show the same graphs but for values of Δx closer to 0. They illustrate the difference, respectively, between the linear convergence of the right and left Riemann sums, and the quadratic convergence of the midpoint Riemann sum and the trapezoid rule.

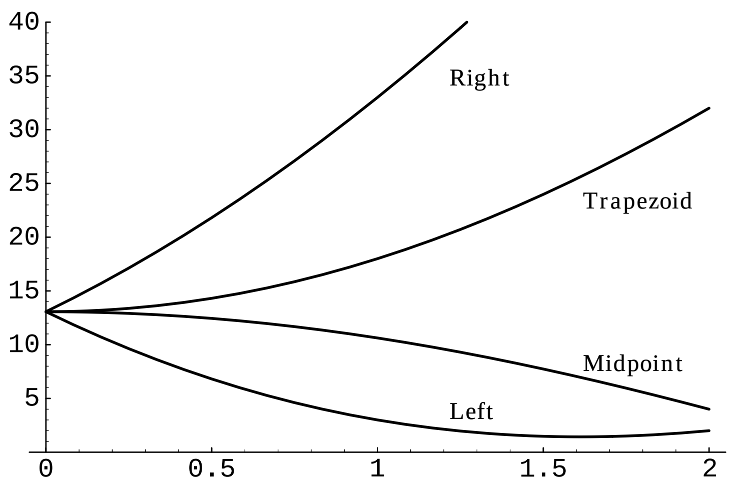

FIGURE 2
The right, left and midpoint Riemann sums, and the trapezoid approximations, as functions of Δx.

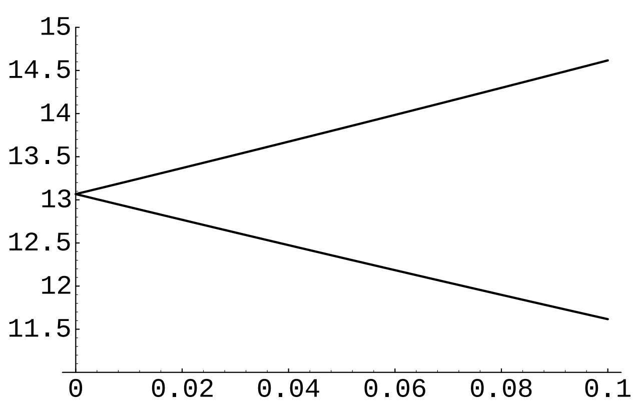

FIGURE 3
The linear convergence of the right and left Riemann sums.

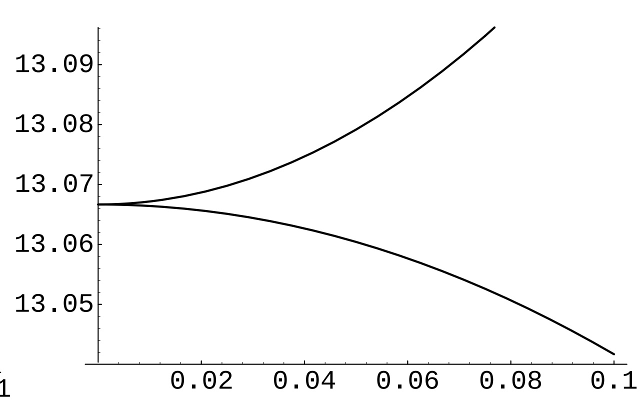

FIGURE 4
The quadratic convergence of the midpoint sum and the trapezoid rule.

As a final investigation with these sums, we talk about how one might eliminate the quadratic term from the approximation for the integral. It is clear in this example that a weighted average involving twice the midpoint expression plus the trapezoid expression should be used, giving Simpson's rule in the form

$$\mathrm{Simp}(\Delta x) = \frac{2\,\mathrm{Mid}(\Delta x) + \mathrm{Trap}(\Delta x)}{3} = \frac{196}{15} + \frac{\Delta x^4}{60}.$$

This version of Simpson's rule differs from the classical one, however, in the meaning of n and Δx. For example, when $\Delta x = 0.40$ there would be five intervals for the midpoint and trapezoid rules. In this case, Simp(0.40) would be the approximation obtained by using the left and right endpoints and the midpoint of five intervals of width Δx, and hence would correspond to ten intervals of width 0.20 in the classical Simpson's rule. In Figure 5 we add the graph of Simp(Δx) to the previous ones and notice the tremendous improvement in the rate of convergence of the Simpson approximation. This is due to the fact that the convergence is now controlled by the fourth power of Δx.

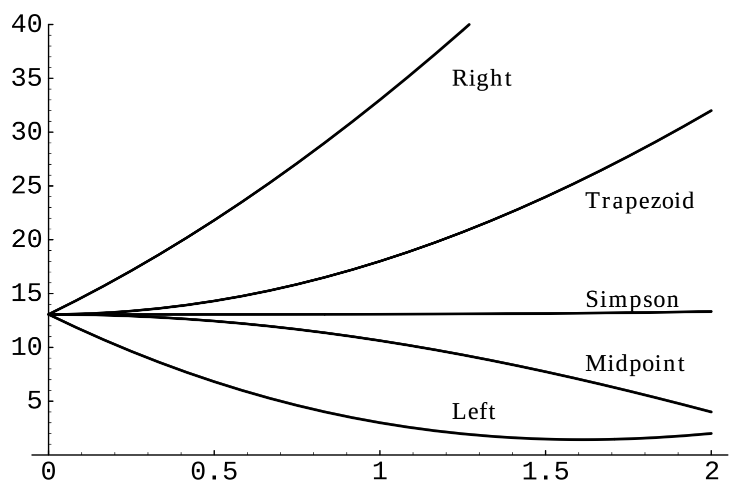

FIGURE 5
All five approximations considered as functions of Δx.

Do these observations hold for the integration of other polynomials? I like to have each student investigate her own polynomial as a homework project, usually constructed using digits from the student's social security number. This provides numerous examples that can be used for classroom discussion and conjectures. Each student is asked to compute the formulas for the approximating sums using left endpoints, right endpoints, midpoints, the trapezoid rule, and Simpson's rule, in the form given above, and then to address the following issues:

1. Use the definition to compute the value of each approximation when using three subintervals. The idea here is to make sure you understand how each approximation

is constructed by doing the calculations yourself. This also provides a check that the symbolic calculations were done correctly.

2. For each sum, compute the limit as Δx goes to 0 and compare this with the value of the definite integral obtained using the antiderivative.
3. Draw the graphs of LEFT(Δx), RIGHT(Δx), MID(Δx), TRAP(Δx), and SIMP(Δx) in the same window. The vertical scale should be chosen so that one can clearly see what is happening to the graphs near $\Delta x = 0$. Discuss how the limits from part (2) are reflected in the graphs. Describe any other interesting aspects of the graphs that relate to your approximations of the definite integral.
4. For each sum, decide whether the error is approximately proportional to Δx, $(\Delta x)^2$, $(\Delta x)^3$, or $(\Delta x)^4$ for small values of Δx. How can you tell?
5. For small values of Δx, are the errors for the left and right rules approximately equal in absolute value and opposite in sign? How can you tell? For small values of Δx, is the error for the midpoint rule about half the size of the trapezoid error and opposite in sign? How can you tell?

It is particularly instructive if there is at least one cubic polynomial among the examples, since Simpson's rule will give the exact value of the definite integral for every value of Δx. Of course, a computer algebra system can evaluate all these sums for a polynomial with symbolic coefficients as well as it can for one with specific numeric coefficients.[2] Such an approach would verify that these properties hold in general but may not provide much insight into the reasons *why*. It might, however, provide motivation for additional study later in a real analysis or numerical analysis course.

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[2]For example, the trapezoid rule for the integral of $Ax^4 + Bx^3 + Cx^2 + Dx + E$ on the interval [0,2] is given by

$$\text{Trap}(\Delta x) = \text{constant} + \left(\frac{8A}{3} + B + \frac{C}{3}\right)\Delta x^2 - \frac{A\,\Delta x^4}{15}$$

and Simpson's rule is

$$\text{Simp}(\Delta x) = \text{constant} + \frac{A\,\Delta x^4}{60}.$$

Example 2. The Sine Function $\int_0^2 \sin(x)\,dx$

This is an example I usually do in the computer lab or assign as a special project. Once again we set up the Riemann sums using n subintervals of equal width Δx on the interval [0, 2] for the cases using right and left endpoints. After evaluating these sums with Mathematica and substituting $2/\Delta x$ for n, we obtain[3]

$$\text{Right}(\Delta x) = \Delta x \csc\left(\frac{\Delta x}{2}\right) \sin(1) \sin\left(1 + \frac{\Delta x}{2}\right),$$

$$\text{Left}(\Delta x) = \Delta x \csc\left(\frac{\Delta x}{2}\right) \sin(1) \sin\left(1 - \frac{\Delta x}{2}\right).$$

It seems clear that each sum will have the same limit as Δx goes to 0 since the only difference between the two is in the last sine function of each expression, but it is perhaps not as clear what the value of this limit will be. Of interest, however, is the observation that almost the same expression, $\sin(x)/x$, is involved as is found in determining the derivative of the sine function from the difference quotient. This problem therefore gives a nice application of some of the ideas about limits and leads to the conclusion that both of the Riemann sums converge to $2\sin^2(1)$.[4]

What about the Riemann sums using points other than at the endpoints of the subintervals? Using the same parameter s as in Example 1, we find that the sum using the point at a distance $s\Delta x$ from the right endpoint of each subinterval is given by

$$\sum_{k=1}^{n} \sin\big((k - s)\Delta x\big)\Delta x = \Delta x \csc\left(\frac{\Delta x}{2}\right) \sin(1) \sin\left(1 + \frac{\Delta x}{2} - s\Delta x\right).$$

Again we see that the sum converges to $2\sin^2(1)$ as Δx goes to 0 for every choice of s. Moreover, the choice of $s = 1/2$ simplifies the last part of this expression and gives the midpoint Riemann sum as

$$\text{Mid}(\Delta x) = \Delta x \csc\left(\frac{\Delta x}{2}\right) \sin^2(1).$$

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[3]Derive, version 2.50, gives the answer

$$\text{Right}(\Delta x) = \frac{\Delta x \cot(\Delta x)}{2} + \frac{\Delta x}{2\sin(\Delta x)} - \frac{\Delta x \cos(\Delta x/2 + 2)}{2\sin(\Delta x/2)}.$$

The Student Edition of Maple V, Release 2, gives

$$\text{Right}(\Delta x) = \frac{\Delta x/2\,\big(-\sin(2 + \Delta x)\cos(\Delta x) + \sin(2 + \Delta x) + \sin(\Delta x)\cos(2 + \Delta x) - \sin(\Delta x)\big)}{\cos(\Delta x) - 1}.$$

[4]The result from using antiderivatives would give $1 - \cos(2)$. It is an easy use of trigonometric identities to show that these two expressions are the same.

The trapezoid rule for this integral simplifies to

$$\mathrm{Trap}(\Delta x) = \frac{\mathrm{Right}(\Delta x) + \mathrm{Left}(\Delta x)}{2} = \Delta x \cot\left(\frac{\Delta x}{2}\right) \sin^2(1).$$

The reader is invited to graph the three Riemann sums and the trapezoid approximation considered as functions of Δx to notice that the behavior of the convergence to the limit as Δx goes to 0 looks very similar to that in Figure 2. An interesting discussion for the students is whether the relative positions of the curves in the graph of the respective sums versus Δx are reasonable in light of the behavior of the function under investigation.

A more precise analysis of the rates of convergence of these approximations provides an application of Taylor series for those students who have covered this material. For example, using Mathematica to compute the Taylor series at 0 shows that

$$\mathrm{Mid}(\Delta x) = 2\sin^2(1) + \frac{\sin^2(1)}{12}\Delta x^2 + \frac{7\sin^2(1)}{2880}\Delta x^4 + O(\Delta x^6)$$

and

$$\mathrm{Trap}(\Delta x) = 2\sin^2(1) - \frac{\sin^2(1)}{6}\Delta x^2 - \frac{\sin^2(1)}{360}\Delta x^4 + O(\Delta x^6).$$

Again, notice that Simpson’s rule will eliminate the Δx^2 terms and produce an approximation that has a rate of convergence controlled by the fourth power of Δx.

Example 3. Double Integrals $\int_0^1\int_0^2 (x^3y^2 - 2x^2 + xy + 1)\,dy\,dx$

I like to return to this topic when introducing multiple integrals. Again, this is usually done in a laboratory setting with students working on a common function, followed later by individual projects. We take as our definition of the double integral the one found in [2, p.281]. For a function f of two variables, we construct a double Riemann sum of the form

$$\sum_{h=1}^{n}\sum_{k=1}^{n} f(c_{kh})\Delta x\,\Delta y$$

over a rectangle $R = [a, b] \times [c, d]$ by partitioning each of the intervals $[a, b]$ and $[c, d]$ into n subintervals of width $\Delta x = (b - a)/n$ and $\Delta y = (d - c)/n$, respectively, and choosing a point c_{kh} in each of the n^2 subrectangles. The function is integrable if the sequence of sums so constructed converges to a limit S as $n \to \infty$ and the limit S is the same for any choice of points c_{kh}. This is not the most general definition of the double integral but is one that works well for investigations with computer algebra systems.

There are several strategies the students can use to pick the points c_{kh} for this particular example on the rectangle $[0, 1] \times [0, 2]$, some of which are illustrated in Figure 6. To reduce the amount of computations, I have each student pick just one corner point to use for the Riemann sum. For example, if we use the upper right corner point of each subrectangle, then the double Riemann sum evaluates to

$$\sum_{h=1}^{n}\sum_{k=1}^{n} f(k\Delta x, h\Delta y)\Delta x\Delta y = \Delta x\Delta y\, n^2[24 - 8\Delta x^2 + 6\Delta x\Delta y - 24\Delta x^2 n + 12\Delta x\Delta y\, n$$
$$+ \Delta x^3\Delta y^2 n - 16\Delta x^2 n^2 + 6\Delta x\Delta y\, n^2 + 5\Delta x^3\Delta y^2 n^2 + 9\Delta x^3\Delta y^2 n^3$$
$$+ 7\Delta x^3\Delta y^2 n^4 + 2\Delta x^3\Delta y^2 n^5] \,/\, 24\,.$$

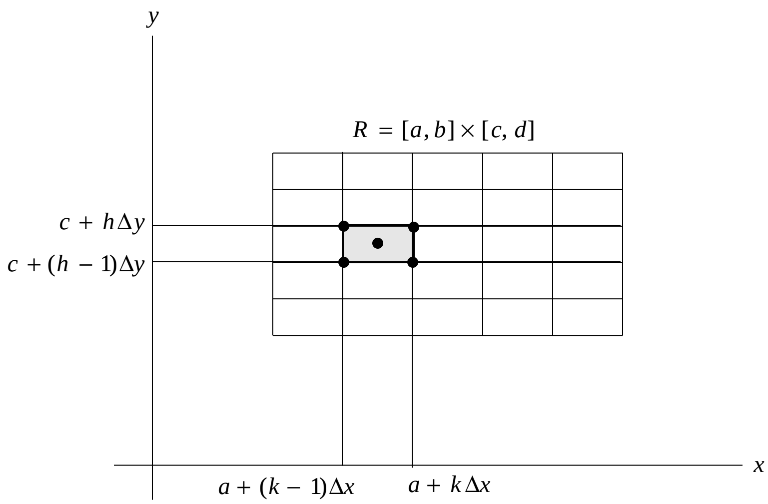

FIGURE 6
Partition of R and possible choices for where to evaluate the function.

It is convenient to express this sum entirely in terms of Δx. Substituting $n = 1/\Delta x$ and $\Delta y = 2\Delta x$ gives the sum for the upper right corner as

$$\mathrm{UR}(\Delta x) = \frac{7}{3} + \frac{7}{3}\Delta x + \frac{10}{3}\Delta x^2 + \frac{5}{3}\Delta x^3 + \frac{1}{3}\Delta x^4.$$

In a similar way we obtain for each of the choices of the other three corners the following double Riemann sums:

(upper left)

$$\mathrm{UL}(\Delta x) = \sum_{h=1}^{n}\sum_{k=1}^{n} f\big((k-1)\Delta x, h\Delta y\big)\Delta x\Delta y = \frac{7}{3} + \frac{5}{3}\Delta x - \frac{8}{3}\Delta x^2 + \frac{1}{3}\Delta x^3 + \frac{1}{3}\Delta x^4$$

(lower left)

$$\mathrm{LL}(\Delta x) = \sum_{h=1}^{n} \sum_{k=1}^{n} f\big((k-1)\Delta x, (h-1)\Delta y\big)\Delta x\,\Delta y = \frac{7}{3} - \frac{7}{3}\Delta x + \frac{10}{3}\Delta x^2 - \frac{5}{3}\Delta x^3 + \frac{1}{3}\Delta x^4$$

(lower right)

$$\mathrm{LR}(\Delta x) = \sum_{h=1}^{n} \sum_{k=1}^{n} f\big(k\Delta x, (h-1)\Delta y\big)\Delta x\,\Delta y = \frac{7}{3} - \frac{5}{3}\Delta x - \frac{8}{3}\Delta x^2 - \frac{1}{3}\Delta x^3 + \frac{1}{3}\Delta x^4$$

There are several observations I try to get the students to make from these results. First, in all four cases, the limit as $\Delta x \to 0$ is equal to 7/3. Second, the sums converge to this limit at a (slow) linear rate governed by the Δx term for values of Δx near 0 (i.e., for large values of n.) This is illustrated graphically in Figure 7 where $\mathrm{UR}(\Delta x)$, $\mathrm{UL}(\Delta x)$, $\mathrm{LL}(\Delta x)$, and $\mathrm{LR}(\Delta x)$ are plotted for values of Δx near 0. Finally, averaging the four Riemann sums produces a new approximation that converges to 7/3 at a quadratic rate governed by a Δx^2 term for small values of Δx, and also eliminates the Δx^3 term.

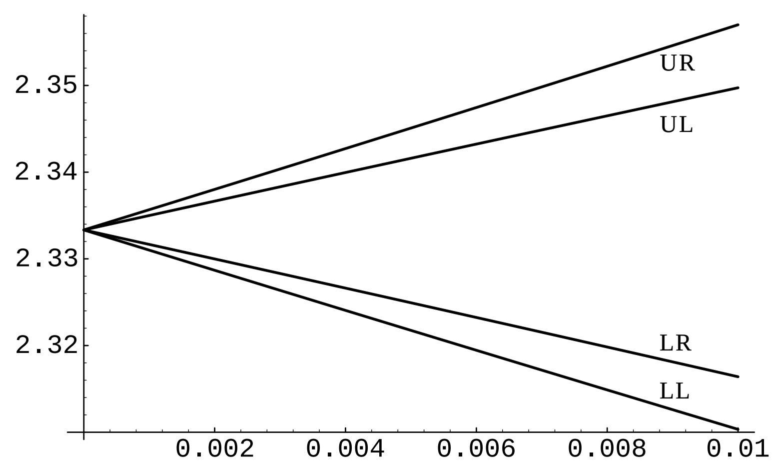

Figure 7
The Riemann sums using each of the possible corner points, considered as functions of Δx.

There are many other questions about double Riemann sums that students can investigate with the aid of a computer algebra system. For example, what happens for this example with an "arbitrary" double sum of the form

$$\sum_{h=1}^{n} \sum_{k=1}^{n} f\big((k-s)\Delta x, (h-t)\Delta y\big)\Delta x\,\Delta y?$$

Does this sum converge to 7/3 for all choices of s and t? What choices of s and t will result in a double Riemann sum that has a quadratic rate of convergence? Is there a choice of s and t that will eliminate both the Δx term and the Δx^3 term?[5] If so, how can this double Riemann sum and the average of the four Riemann sums given above be combined to improve the rate of convergence, that is, what is the corresponding Simpson's rule for functions of two variables?

Conclusion

A computer algebra system allows students to experiment with the concept of the Riemann sum in a way that is not possible with just numerical calculations (see [1], [3] and [4] for similar approaches.) The results demonstrate rather convincingly that these sums do converge to the same value for the definite integral that is obtained using antiderivatives and the Fundamental Theorem. Moreover, the investigations about rates of convergence illustrate approximation techniques that students can exploit when having to integrate functions represented only by tabular data rather than by explicit formulas. Finally, students can investigate many interesting examples in addition to the ones mentioned in this article, such as the Riemann sums of $x^2 \sin(x)$, a function that ordinarily would require integration by parts[6], or the double Riemann sums for $\sin(x + y)$ over a rectangle, or even triple Riemann sums for functions of three variables defined over a box in 3-space.

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[5] Only the choice of the center point of each subrectangle ($s = t = 1/2$) will accomplish this goal. This is another nice application of the CAS.

[6] Derive cannot simplify the Riemann sums for this function.

Appendix 1

We show here why the trapezoid and midpoint rules for polynomials only involve even powers of Δx. The results are easily extended to other functions represented by Taylor series.

Let P be a polynomial of degree r. Divide the interval $[a, b]$ into n subintervals of width $\Delta x = (b - a)/n$. Let c_k represent the midpoint of the kth subinterval. Then on the kth subinterval

$$P(a + (k - 1)\Delta x) + P(a + k\Delta x) = P\left(c_k - \frac{\Delta x}{2}\right) + P\left(c_k + \frac{\Delta x}{2}\right)$$

$$= \sum_{i=0}^{r} \frac{P^{(i)}(c_k)}{i\,!}\left(-\frac{\Delta x}{2}\right)^i + \sum_{i=0}^{r} \frac{P^{(i)}(c_k)}{i\,!}\left(\frac{\Delta x}{2}\right)^i$$

$$= \sum_{i=0}^{r} \frac{P^{(i)}(c_k)}{i\,!}\left[\left(-\frac{\Delta x}{2}\right)^i + \left(\frac{\Delta x}{2}\right)^i\right].$$

For each value of i there is, by the Intermediate Value Theorem, a value x_i in the interval $[a, b]$ such that

$$\frac{1}{n}\sum_{k=1}^{n} P^{(i)}(c_k) = P^{(i)}(x_i).$$

Now the trapezoid rule can be written as

$$\mathrm{Trap}(\Delta x) = \frac{\Delta x}{2}\sum_{k=1}^{n}\left[P(a + (k - 1)\Delta x) + P(a + k\Delta x)\right]$$

$$= \frac{\Delta x}{2}\sum_{k=1}^{n}\sum_{i=0}^{r} \frac{P^{(i)}(c_k)}{i\,!}\left[\left(-\frac{\Delta x}{2}\right)^i + \left(\frac{\Delta x}{2}\right)^i\right]$$

$$= \frac{\Delta x}{2}\sum_{i=0}^{r} \frac{1}{i\,!}\left[\sum_{k=1}^{n} P^{(i)}(c_k)\right]\left[\left(-\frac{\Delta x}{2}\right)^i + \left(\frac{\Delta x}{2}\right)^i\right]$$

$$= \frac{\Delta x}{2}\sum_{i=0}^{r} \frac{1}{i\,!}\left[nP^{(i)}(x_i)\right]\left[\left(-\frac{\Delta x}{2}\right)^i + \left(\frac{\Delta x}{2}\right)^i\right]$$

$$= \frac{b - a}{2}\sum_{i=0}^{r} \frac{P^{(i)}(x_i)}{i\,!}\left[\left(-\frac{\Delta x}{2}\right)^i + \left(\frac{\Delta x}{2}\right)^i\right].$$

The odd values of i result in the cancellation of the Δx terms and therefore only the even powers of Δx remain.

Since $\text{Right}(\Delta x) - \text{Left}(\Delta x) = (P(b) - P(a))\Delta x$, the right and left Riemann sums differ only in the Δx term. On the other hand, the average of the right and left Riemann sums is the trapezoid expression which, as we've just seen, has no odd powers of Δx. Thus neither the right Riemann sum nor the left Riemann sum can contain any odd powers of Δx other than the linear term.

Let F be an antiderivative of the polynomial P. Using the same notation as above, notice that on the kth subinterval

$$\begin{aligned}
P(c_k)\Delta x - \int_{c_k - \frac{\Delta x}{2}}^{c_k + \frac{\Delta x}{2}} P(x)\,dx &= P(c_k)\Delta x - \left[F\left(c_k + \frac{\Delta x}{2}\right) - F\left(c_k - \frac{\Delta x}{2}\right)\right] \\
&= P(c_k)\Delta x - \sum_{i=0}^{r} \frac{F^{(i)}(c_k)}{i\,!}\left[\left(\frac{\Delta x}{2}\right)^i - \left(-\frac{\Delta x}{2}\right)^i\right] \\
&= \sum_{i=3}^{r} \frac{F^{(i)}(c_k)}{i\,!}\left[\left(\frac{\Delta x}{2}\right)^i - \left(-\frac{\Delta x}{2}\right)^i\right].
\end{aligned}$$

Using the Intermediate Value Theorem on the function F, we now get

$$\begin{aligned}
\text{Mid}(\Delta x) - \int_a^b P(x)\,dx &= \sum_{k=1}^{n}\left[P(c_k)\Delta x - \int_{c_k - \frac{\Delta x}{2}}^{c_k + \frac{\Delta x}{2}} P(x)\,dx\right] \\
&= \sum_{k=1}^{n}\sum_{i=3}^{r} \frac{F^{(i)}(c_k)}{i\,!}\left[\left(\frac{\Delta x}{2}\right)^i - \left(-\frac{\Delta x}{2}\right)^i\right] \\
&= \sum_{i=3}^{r} \frac{1}{i\,!}\left[\sum_{k=1}^{n} F^{(i)}(c_k)\right]\left[\left(\frac{\Delta x}{2}\right)^i - \left(-\frac{\Delta x}{2}\right)^i\right] \\
&= \sum_{i=3}^{r} \frac{1}{i\,!}\, nF^{(i)}(w_i)\left[\left(\frac{\Delta x}{2}\right)^i - \left(-\frac{\Delta x}{2}\right)^i\right] \\
&= (b-a)\sum_{i=3}^{r} \frac{1}{i\,!}F^{(i)}(w_i)\left[\left(\frac{\Delta x}{2}\right)^i - \left(-\frac{\Delta x}{2}\right)^i\right]\frac{1}{\Delta x}.
\end{aligned}$$

This last expression contains only even powers of Δx corresponding to the odd values of i.