

Branch Tips for Golden 108 Symmetric Binary Trees

$$\begin{aligned}
 k &= \cos(18) + i \sin(18) \\
 k &:= \text{sqrt}\left(\frac{5 + \text{sqrt}(5)}{8}\right) + I \cdot \frac{\text{sqrt}(5) - 1}{4} \\
 &\quad \frac{1}{4} \sqrt{10 + 2\sqrt{5}} + I \left(\frac{1}{4} \sqrt{5} - \frac{1}{4}\right)
 \end{aligned} \tag{1}$$

$$\begin{aligned}
 S &= \sqrt{5 - \sqrt{5}} (\cos(36) - i \sin(36)) \\
 S &:= \text{evalc}\left(\text{factor}\left(\text{expand}\left(\text{sqrt}(5 - 2 \text{sqrt}(5)) \cdot \left(\frac{1 + \text{sqrt}(5)}{4} - I \cdot \text{sqrt}\left(\frac{5 - \text{sqrt}(5)}{8}\right)\right)\right)\right)\right) \\
 &\quad \frac{1}{4} \sqrt{5 - 2\sqrt{5}} + \frac{1}{4} \sqrt{5 - 2\sqrt{5}} \sqrt{5} + I \left(-\frac{3}{4} \sqrt{5} + \frac{5}{4}\right)
 \end{aligned} \tag{2}$$

$$\begin{aligned}
 \alpha &= \cos(108) + i \sin(108) = -\sin(18) + i \cos(18) \\
 \alpha &:= \frac{1 - \text{sqrt}(5)}{4} + I \cdot \text{sqrt}\left(\frac{5 + \text{sqrt}(5)}{8}\right) \\
 &\quad \frac{1}{4} - \frac{1}{4} \sqrt{5} + \frac{1}{4} I \sqrt{10 + 2\sqrt{5}}
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 r &:= \frac{\text{sqrt}(5) - 1}{2} \\
 &\quad \frac{1}{2} \sqrt{5} - \frac{1}{2}
 \end{aligned} \tag{4}$$

Green tree branch tip

$$\begin{aligned}
 \text{greenBP} &:= \text{evalc}\left(\text{simplify}\left(\text{evalc}\left(I + I \cdot \alpha^{-1} + I \cdot r \cdot \alpha^{-2}\right)\right)\right) \\
 &\quad -\frac{1}{8} \sqrt{10 + 2\sqrt{5}} + \frac{1}{8} \sqrt{5} \sqrt{10 + 2\sqrt{5}} + I \left(\frac{3}{4} - \frac{1}{4} \sqrt{5}\right)
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 \text{evalf}(\text{greenBP}) \\
 &\quad 0.5877852519 + 0.1909830058 I
 \end{aligned} \tag{6}$$

Red tree branch tip

$$\begin{aligned}
 \text{redBP} &:= \text{evalc}\left(\text{simplify}\left(\text{evalc}\left(S + k + k \cdot \alpha + k \cdot r \cdot \alpha^2\right)\right)\right) \\
 &\quad \frac{1}{4} \sqrt{5 - 2\sqrt{5}} + \frac{1}{4} \sqrt{5 - 2\sqrt{5}} \sqrt{5} + I \left(\frac{3}{4} - \frac{1}{4} \sqrt{5}\right)
 \end{aligned} \tag{7}$$

$$\begin{aligned}
 \text{evalf}(\text{redBP}) \\
 &\quad 0.5877852527 + 0.1909830058 I
 \end{aligned} \tag{8}$$

We see that the two branch points are the same numerically. To verify this symbolically, subtract the two values and simplify.

$$\begin{aligned}
 \text{greenBP} - \text{redBP} \\
 &\quad -\frac{1}{8} \sqrt{10 + 2\sqrt{5}} + \frac{1}{8} \sqrt{5} \sqrt{10 + 2\sqrt{5}} - \frac{1}{4} \sqrt{5 - 2\sqrt{5}} - \frac{1}{4} \sqrt{5 - 2\sqrt{5}} \sqrt{5}
 \end{aligned} \tag{9}$$

$$\begin{aligned}
 \text{simplify}(\%) \\
 &\quad 0
 \end{aligned} \tag{10}$$

