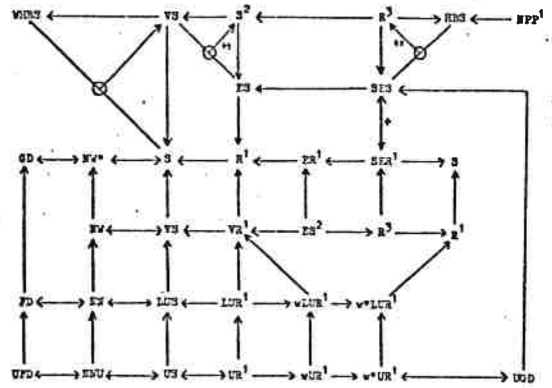

THE

GEOMETRY OF BANACH SPACES

AND

REFLEXIVITY

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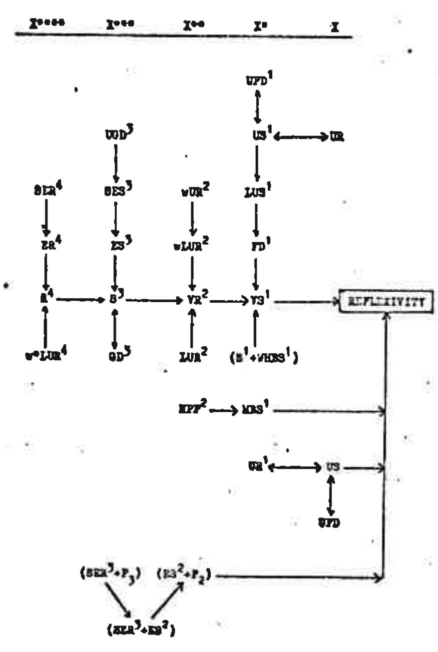

THE GEOMETRY OF BANACH SPACES AND REFLEXIVITY

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1. INTRODUCTION

Ever since 1927, when Hans Hahn [31] introduced the concept of a regular space, later to be called reflexive by E.R.Lorch [38], mathematicians have sought sufficient conditions for a Banach space to possess this property. In this paper we investigate some geometrical properties of a Banach space and its conjugate spaces, and determine those which imply reflexivity. We display these implications graphically in a form modelled from a paper of Francis Sullivan [57].

Throughout this paper X will always denote a real Banach space and X^*, X^{**}, X^{***}, and $X^{(4)}$ its successive conjugate spaces. We denote by Q_0, Q_1, and Q_2 the canonical embeddings of X, X^*, and X^{**} into X^{**}, X^{***}, and $X^{(4)}$ respectively, and by S and B the unit sphere and unit ball of X, with the obvious extension of notation to the conjugate spaces. When no confusion can result we consider a space as a subspace of its second conjugate, and write, for example, $x \in X^{**}$ to mean $Q_0(x) \in X^{**}$. It is well known that $X^{***} = X^* \oplus X^{\perp}$, where $X^{\perp}$ denotes the annihilator of $Q_0(X)$ in X^{***}. X^* and $X^{\perp}$ are the range and null space respectively of the norm-1 projection $Q_1 Q_0{}^*$.

E. Bishop and R. R. Phelps have shown [3] that every Banach space Y is subreflexive, i.e. the collection of functionals which attain their norm on the unit sphere of Y in norm-dense in Y^*. James [34] expanded upon this result by proving that a Banach space Y is reflexive if and only if this collection of functionals actually comprises all of Y^*. Both of these theorems will be used extensively.

Goldstine's theorem says that $Q_0(B)$ is weak*-dense in

B** [32]. Unless X* is separable, however, this only implies the existence of nets converging weak* to a given element, while we shall usually want a sequence. Fortunately, Lindenstruass and Rosenthal [37] obtained a stronger version of Goldstine's theorem which is called the principle of local reflexivity. The form we shall need is due to Dean [13]: If $A \subset X^{**}$ and $F \subset X^*$ are finite dimensional subspaces and $0 < \delta < 1$ is arbitrary, then there exists a linear map $T: A \to X$ such that

(a) $T(a) = a$ for all $a \in A \cap X$

(b) $f(T(a)) = a(f)$ for all $a \in A$ and $f \in F$

(c) $(1-\delta)||a|| \leq ||T(a)|| \leq (1+\delta)||a||$ for all $a \in A$.

A mapping $x \mapsto f_x$ from $X \setminus \{0\}$ to $X^* \setminus \{0\}$ is called a <u>support mapping</u> if

(i) whenever $x \in S$, then $||f_x|| = 1 = f_x(x)$;

(ii) for every $\beta > 0$, $f_{\beta x} = \beta f_x$.

Support mappings always exist, for by the Hahn-Banach theorem we may define a mapping on S so as to satisfy (i) and then extend it to $X \setminus \{0\}$ using (ii). A functional $f \in S^*$ satisfying $f(x) = 1$ for $x \in S$ is called a <u>support functional</u> of x in X*. Then $f^{<}(\{1\})$ (the inverse image of 1 under f) defines a "tangent" hyperplane to S at x.

By considering the cases $\beta > 0$ and $\beta < 0$ separately, a straightfoward calculation yields the following useful inequality; for a support mapping $x \mapsto f_x$ and for every $x,y \in S$ and $\beta \neq 0$,

$$(1) \qquad \left| \frac{||x+\beta y|| - 1}{\beta} - f_x(y) \right| \leq \left| \frac{f_{x+\beta y}(y)}{||x+\beta y||} - \frac{f_x(y)}{||x||} \right| .$$

This will be used to provide connections between continuity properties of support mappings and differentiability conditions on the norm of the Banach space. We list below the relevant definitions: consider

(2) $$\lim_{\beta\to 0} \frac{||x+\beta y|| - 1}{\beta}$$

(i) X is said to be <u>Gateaux differentiable</u> if for each $x,y \in S$ the limit in (2) exists;

(ii) X is said to be <u>Fréchet differentiable</u> if for each $x \in S$ the limit in (2) exists uniformly in $y \in S$;

(iii) X is said to be <u>uniformly Gateaux differentiable</u> if for each $y \in S$ the limit in (2) exists uniformly in $x \in S$;

(iv) X is said to be <u>uniformly Fréchet differentiable</u> if the limit in (2) exists uniformly in $x,y \in S$.

For $x,y \in S$, let $p(x,y) := ||x+y|| + ||x-y|| - 2$ (*). Then the limit in (2) exists if and only if

(3) $$\lim_{\beta\to 0} \frac{p(x,\beta y)}{\beta} = 0,$$

with any uniformity conditions being preserved.

Finally, the reader is referred to the bibliography for a listing of the literature from which all of the cited definitions and results were obtained.

(*) In a formal definition by means of an equality, the colon next to the equality sign specifies the definiendum.

2. THE GEOMETRY OF BANACH SPACES

2.1 Smoothness and rotundity

The Hahn-Banach theorem guarantees the existence of at least one support mapping. The first property we consider says that this mapping is unique.

DEFINITION: X is said to be smooth if for each $x \in S$ there exists a unique element $x^* \in S^*$ such that $x^*(x) = 1$.

In a smooth space each point on the unit sphere thus has a unique tangent hyperplane.

PROPOSITION 2.1.1: The following are equivalent:

(i) X is smooth;

(ii) Every support mapping is norm-to-weak* continuous from S to S*;

(iii) There exists a support mapping which is norm-to-weak* continuous from S to S*;

(iv) X is Gateaux differentiable.

Proof. (i) implies (ii): Suppose $x \mapsto f_x$ is a support mapping which is not norm-to-weak* continuous at some point $x \in S$. Then there exists a sequence (x_n) in S such that $||x_n - x|| \to 0$ but $f_n := f_{x_n}$ does not converge weak* to f_x. By passing to a subsequence if necessary, we may assume without loss of generality that there exists a weak* neighborhood U of f_x such that

(4) $$f_n \notin U \quad (n \in \mathbb{N}).$$

Now B* is weak*-compact, so (f_n) has a weak*-cluster point $x^* \in B^*$. Since

$$\begin{aligned} |x^*(x) - 1| &= |x^*(x) - f_n(x_n)| \\ &\leq |x^*(x) - f_n(x)| + |f_n(x) - f_n(x_n)| \\ &\leq |x^*(x) - f_n(x)| + ||x - x_n||, \end{aligned}$$

and the expression on the right can be made arbitrarily small, we must have $x^*(x) = 1$. Therefore $||x^*|| = 1$. But by (4) $x^* \neq f_x$ so that x has two distinct support functionals in X^*. Therefore X is not smooth.

(ii) implies (iii) is trivial.

(iii) implies (iv) follows from inequality (1) since

$$\left|\left| \frac{x + \beta y}{||x+\beta y||} - \frac{x}{||x||} \right|\right| \to 0$$

as $\beta \to 0$.

(iv) implies (i): A straightfoward calculation shows that for a support mapping $x \mapsto f_x$, if $-1 < \alpha < 0 < \beta < 1$ and $x, y \in S$, then

$$\frac{||x+\alpha y|| - 1}{\alpha} \leq f_x(y) \leq \frac{||x+\beta y|| - 1}{\beta} .$$

Thus if X is Gateaux differentiable, we must have

$$f_x(y) = \lim_{\beta \to 0} \frac{||x+\beta y|| - 1}{\beta} ,$$

and so every support functional for x agrees on S. Thus there can be only one such functional for each $x \in S$, whence we conclude that X is smooth. ▨

<u>COROLLARY 2.1.2</u>: X is smooth if and only if for each $x, y \in S$,

$$\lim_{\beta \to 0} \frac{p(x, \beta y)}{\beta} = 0.$$

We now look at support functionals from another direction. Instead of asking which elements of S possess a unique support functional, we ask which functionals in S^* are support functionals for a unique element of S.

<u>DEFINITION</u>: X is said to be <u>rotund</u> (or strictly

convex) if no $x^* \in S^*$ is simultaneously a support functional for some $x \in S$ and $y \in S$, where $x \neq y$.

Of course, for $x^* \in S^*$ there may not exist any element $x \in S$ at which x^* attains its norm. Indeed, by James' theorem this certainly happens for some functional if X is not reflexive. What we require for rotundity is that <u>if</u> x exists, then it be unique.

The following results give several characterizations of rotundity.

<u>PROPOSITION 2.1.3</u>: X is rotund if and only if S contains no line segment, i.e. for $x,y \in S$, if $||x+y|| = 2$, then $x = y$.

Proof. Suppose X is not rotund. Then there exists $x^* \in S^*$ and $x,y \in S$ such that $x^*(x) = 1 = x^*(y)$ and $x \neq y$. Then

$$2 = x^*(x+y) \leq ||x+y|| \leq 2,$$

and so $||x+y|| = 2$.

Conversely, suppose $||x|| = ||y|| = \left|\left|\frac{x+y}{2}\right|\right| = 1$, but $x \neq y$. Let $x^* \in S^*$ be a support functional for $(x+y)/2$. Then $x^*(x) + x^*(y) = 2$ and since $x^*(x) \leq 1$, $x^*(y) \leq 1$, we must have $x^*(x) = 1 = x^*(y)$. Thus X is not rotund. ▨

Thus X is rotund if and only if for $x,y \in X$, if $||x+y|| = ||x|| + ||y||$ then x and y are linearly dependent.

<u>DEFINITION</u>: A <u>duality mapping</u> for X is the function $J:X \to \mathcal{P}(X^*)$ given by

$$J(x) := \{x^* \in X^*: x^*(x) = ||x^*||\,||x||,\ ||x^*|| = ||x||\}.$$

J is said to be <u>strictly monotone</u> if for every distinct $x,y \in X$ and every $x^* \in J(x)$ and $y^* \in J(y)$ we have $(x^*-y^*)(x-y) > 0$.

For $x^* \in J(x)$ and $y^* \in J(y)$, Browder observed that

(7) $$(x^*-y^*)(x-y) = (||x||-||y||)^2 + (||x^*||\,||y||-x^*(y))$$

$$+ (||y^*|| \, ||x|| - y^*(x))$$

$$\geq 0.$$

PROPOSITION 2.1.4 (Petryshyn): X is rotund if and only if J is strictly monotone.

Proof. Since each term in (7) is non-negative, J is not strictly monotone if and only if there exists $x \neq y$ and $x^* \in J(x)$, $y^* \in J(y)$ such that

$$\text{(8)} \qquad \begin{aligned} ||x|| &= ||y|| \\ ||x^*|| \, ||y|| &= x^*(y) \\ ||y^*|| \, ||x|| &= y^*(x). \end{aligned}$$

If this occurs then $x \neq 0 \neq y$ and

$$\frac{x^*}{||x^*||}\left(\frac{y}{||y||}\right) = 1 = \frac{x^*}{||x^*||}\left(\frac{x}{||x||}\right),$$

so X is not rotund. Conversely, if X is not rotund and $x^* \in S^*$ is a support functional for distinct $x, y \in S$, then (8) is satisfied with $y^* := x^*$. ▨

A Banach space might not possess a certain geometrical property under its given norm, yet may under an equivalent one. For example, in the plane both the unit circle and unit square are unit "spheres" of equivalent norms, but the former is rotund while the latter is not. The following result due to V. Klee is an example of when a space may be so renormed.

PROPOSITION 2.1.5: A Banach space Y has an equivalent rotund norm if and only if there exists a rotund Banach space W and an injective continuous linear map $T: Y \to W$.

Proof. If Y already admits an equivalent rotund norm, then Y under this norm and T the identity map satisfy the conditions of the theorem.

Conversely, suppose W is a rotund space and $T: Y \to W$

is an injective continuous linear map. For $y \in Y$ define

$$|||y||| := ||y|| + ||Ty||.$$

Then $|||\cdot|||$ is a norm on Y which is equivalent to $||\cdot||$ since

$$||y|| \leq |||y||| \leq (1+||T||)||y||.$$

If x and y are linearly independent elements of Y, then Tx and Ty are linearly independent in W. Since W is rotund,

$$\begin{aligned} |||x+y||| &= ||x+y|| + ||Tx+Ty|| \\ &\leq ||x|| + ||y|| + ||Tx+Ty|| \\ &< ||x|| + ||y|| + ||Tx|| + ||Ty|| \\ &= |||x||| + |||y|||. \end{aligned}$$

Hence Y under $|||\cdot|||$ is rotund. ▨

In [14] Day showed that c_0 admits an equivalent rotund norm $|||\cdot|||$. Using this result and proposition 2.1.5 it is a simple matter to show that l_∞ admits an equivalent rotund norm, for we just define $T:l_\infty \rightarrow (c_0, |||\cdot|||)$ by

$$(Tx)_n := \frac{x_n}{n}.$$

Then T is clearly linear, continuous and injective.

Smoothness and rotundity are partially dual properties, as shown by

PROPOSITION 2.1.6

(i) If X^* is rotund, then X is smooth.

(ii) If X^* is smooth, then X is rotund.

Proof. (i) Suppose for $x \in S$ we have

$$x^*(x) = 1 = y^*(x)$$

for $x^*, y^* \in S^*$. Then $||x^*+y^*|| = 2$, so by the rotundity of X^*, $x^* = y^*$. Hence X is smooth.

(ii) Suppose for $x^* \in S^*$ we have

$$x^*(x) = 1 = x^*(y)$$

for $x, y \in S$. Then $Q_0(x)(x^*) = 1 = Q_0(y)(x^*)$, so by the smoothness of X^*, $Q_0(x) = Q_0(y)$, i.e. $x = y$. Hence X is rotund. ▨

Day actually showed that $c_0(\Gamma)$ admits an equivalent rotund norm for any index set Γ. However, if Γ is uncountable then $c_0(\Gamma)$ has a non-smooth dual space.

For reflexive spaces we have complete duality between smoothness and rotundity.

2.2 Very smooth and very rotund

The second conjugate space of X will be smooth on S^{**} if and only if each element of S^{**} has a unique support functional in S^{***}. We may formulate a weaker condition by only requiring X^{**} to be smooth on $Q_0(S)$. This property was introduced in [20] in a different (though equivalent) form.

DEFINITION: X is said to be very smooth if for each $x \in S$ there exists a unique element $x^{***} \in S^{***}$ such that $x^{***}(Q_0(x)) = 1$.

By corollary 2.1.2 X is very smooth if and only if for each $x \in S$ and $y^{**} \in S^{**}$

$$\lim_{\beta \to 0} \frac{p(x, \beta y^{**})}{\beta} = 0.$$

A very smooth space is clearly smooth. However, under an equivalent norm c_0 is Fréchet differentiable [16] and hence (by a later result) very smooth, but l_∞ has no equivalent smooth norm [14], so that X very smooth is strictly weaker than X^{**} smooth.

PROPOSITION 2.2.1: The following are equivalent:

(i) X is very smooth;

(ii) Every support mapping is norm-to-weak continuous from S to S^*;

(iii) There exists a support mapping which is norm-to-weak continuous from S to S*.

Proof. (i) implies (ii): Let $x \mapsto f_x$ be a support mapping. If X is very smooth, then by proposition 2.1.1 this mapping is continuous from the norm topology of S to the weak* topology of S*. But since X is very smooth, we may think of this mapping as from S to S*** with range in $Q_1(S^*)$. Then it is continuous from the norm topology of S to the weak* topology of $Q_1(S^*)$, and thus the weak topology of S*, since Q_1 is a homeomorphism between X* with the weak topology and $Q_1(X^*)$ with the weak* topology.

(ii) implies (iii) is obvious.

(iii) implies (i): Let $x \mapsto f_x$ be a support mapping which is norm-to-weak continuous on S. By proposition 2.1.1 X is smooth. Now suppose that X is not very smooth. Then there exists elements $x \in S$ and $x^{***} \in S^{***}$ such that $f_x \neq x^{***}$ but $f_x(x) = 1 = x^{***}(x)$. Let $y^{**} \in S^{**}$ be chosen so that

$$x^{***}(y^{**}) - Q_1(f_x)(y^{**}) =: \varepsilon > 0$$

By the principle of local reflexivity, there exists for each $n \in \mathbb{N}^x$ a linear map $T_n\text{:span}\{f_x, x^{***}\} \to X^*$ such that

(a) $T_n(f_x) = f_x$

(b) $z^{**}(T_n(x^{***}) = x^{***}(z^{**})$ for $z^{**} \in \{x, y^{**}\}$

(c) $1 - \frac{1}{n} \leq ||T_n(x^{***})|| \leq 1 + \frac{1}{n}$.

Put

$$f_n := \frac{T_n(x^{***})}{||T_n(x^{***})||} .$$

Then

$$f_n(x) = \frac{x^{***}(x)}{||T_n(x^{***})||} = \frac{1}{||T_n(x^{***})||} \to 1 ,$$

so by the smoothness of X and the Bishop-Phelps-Bollobás theorem [4] there exists a sequence (y_n) in S with

(9) $$||y_n - x|| \to 0$$

(10) $$||f_{y_n} - f_n|| \to 0$$

By our assumption, (9) implies that $f_{y_n} \to f_x$ weakly, so by (10) $f_n \to f_x$ weakly. However

$$y^{**}(f_n) - y^{**}(f_x) = \frac{x^{***}(y^{**})}{||T_n(x^{***})||} - y^{**}(f_x) ,$$

whence $y^{**}(f_n) - y^{**}(f_x) \to \varepsilon > 0$. This contradiction shows that X must be very smooth. ▨

As we did with smoothness, we may weaken the rotundity requirements for X^{**} by looking only at elements of $Q_1(S^*)$.

<u>DEFINITION</u> (Sullivan [57]): X is said to be <u>very rotund</u> if no $Q_1(x^*) \in Q_1(S^*)$ is simultaneously a support functional for some $Q_0(x) \in S^{**}$ and $x^{**} \in S^{**}$, where $x^{**} \neq Q_0(x)$.

Thus in a very rotund space, no $x^* \in S^*$ is simultaneously a norming element for some $x \in S$ and $x^{**} \in S^{**}$, where $x^{**} \neq Q_0(x)$. One immediately sees that a very rotund space is therefore rotund.

The partial duality between very smooth and very rotund corresponding to proposition 2.1.6 is given by

<u>PROPOSITION 2.2.2</u>

(i) If X^* is very rotund, then X is very smooth.

(ii) If X^* is very smooth, then X is very rotund.

Proof. (i): If X is not very smooth, there exists an $x \in S$ and distinct elements $x^{***}, y^{***} \in S^{***}$ such that

$x^{***}(x) = 1 = y^{***}(x)$. Let $x^* \in S^*$ be chosen such that $x^*(x) = 1$. Then $Q_1(x^*)(x) = 1$, and either $x^{***} \neq Q_1(x^*)$ or $y^{***} \neq Q_1(x^*)$. Thus X^* is not very rotund.

(ii) If X^* is very smooth, then it is smooth. Thus each $x^* \in S^*$ is a norming element of a unique element of S^{**}, so clearly X is very rotund. ▨

Note that in the proof of (ii) we really needed only the assumption that X^* was smooth to conclude that X was very rotund.

Again we have complete duality if X is reflexive.

2.3 Local uniform smoothness and local uniform rotundity

Suppose X is smooth. Then we know that for each $x \in S$ and $y \in S$,

$$\lim_{\beta \to 0} \frac{p(x,\beta y)}{\beta} = 0 .$$

Thus if $\varepsilon > 0$ and $x,y \in S$, we can find a $\delta(\varepsilon,x,y)$ such that

(11) $\qquad ||x+\beta y|| + ||x-\beta y|| < 2 + \beta\varepsilon$ for $0 < \beta < \delta$.

In this section we investigate when δ can be chosen independently of y so that (11) still holds.

DEFINITION: X is said to be *locally uniformly smooth* whenever given $\varepsilon > 0$ and $x \in S$ there exists $\delta(\varepsilon,x) > 0$ such that if $||y|| < \delta$, then

$$||x+y|| + ||x-y|| < 2 + \varepsilon||y||$$

(equivalently, if $y \in S$, then (11) holds.)

PROPOSITION 2.3.1: The following are equivalent:

(i) X is locally uniformly smooth;

(ii) Every support mapping is norm-to-norm continuous from S to S^*;

(iii) There exists a support mapping which is norm-to-norm continuous from S to S*;

(iv) X is Fréchet differentiable.

Proof. (i) implies (ii): Let $x \mapsto f_x$ be a support mapping which in not norm-to-norm continuous at some point $x \in S$. Then there is a sequence (x_n) in S such that for some $\varepsilon > 0$

(12) $$||x-x_n|| \to 0$$

(13) $$||f_x-f_n|| > 2\varepsilon \quad (n \in \mathbb{N})$$

where $f_n := f_{x_n}$. By (13) there is a sequence (y_n) in S such that

$$(f_n-f_x)(y_n) \geq 2\varepsilon \quad (n \in \mathbb{N}).$$

Since

$$|1-f_n(x)| = |f_n(x_n)-f_n(x)| \leq ||x_n-x||$$

we see that $f_n(x) \to 1$. Let $\beta > 0$. Then there exists an n such that $1 - f_n(x) \leq \varepsilon\beta$.

Note that

$$1 - \beta f_x(y_n) = f_x(x-\beta y_n) \leq ||x-\beta y_n||.$$

Then

$$\begin{aligned} ||x+\beta y_n|| + ||x-\beta y_n|| &\geq f_n(x+\beta y_n) + ||x-\beta y_n|| \\ &= f_n(x) + \beta f_n(y_n) + ||x-\beta y_n|| \\ &\geq 1-(1-f_n(x))+2\varepsilon\beta+\beta f_x(y_n)+||x-\beta y_n|| \\ &\geq 1 - \varepsilon\beta + 2\varepsilon\beta + 1 \\ &= 2 + \varepsilon\beta. \end{aligned}$$

Since β was arbitrary, we can not find a $\delta(x,\varepsilon)$ such that (11) holds for all $y \in S$, and so X is not locally uniformly smooth.

(ii) implies (iii) is obvious.

(iii) implies (iv) follows from inequality (1) since for $x \in S$

$$\left\| \frac{x+\beta y}{\|x+\beta y\|} - \frac{x}{\|x\|} \right\| \to 0$$

uniformly in $y \in S$ as $\beta \to 0$.

(iv) implies (i): If X is Fréchet differentiable, then for $x \in S$

$$\lim_{\beta \to 0} \frac{p(x,\beta y)}{\beta} = 0$$

uniformly in $y \in S$, from which it readily follows that X is locally uniformly smooth. ▨

From propositions 2.2.1 and 2.3.1, we see that a locally uniformly smooth space is very smooth. In [57], F. Sullivan gives an example due to Mark Smith of a Banach space which is very smooth but not Fréchet differentiable, and hence not locally uniformly smooth.

The relationship between the continuity properties of the support mappings and the geometrical properties shows that if X is smooth and weak* convergence and norm convergence [weak convergence] of sequences agree on S*, then X is locally uniformly smooth [very smooth].

<u>DEFINITION</u> (Lovaglia[39]): X is said to be <u>locally uniformly rotund</u> whenever given $\varepsilon > 0$ and $x \in S$ there exists a $\delta(\varepsilon,x) > 0$ such that for $y \in S$, $\|x-y\| \geq \varepsilon$ implies $\|x+y\| \leq 2 - \delta$.

Geometrically, this means that if the midpoint of a variable chord in the unit sphere with one endpoint fixed approaches the boundary of the sphere then the length of the chord approaches zero. Thus, if the midpoint of the chord lies on the unit sphere, the length of the chord must be zero, i.e. a locally uniformly rotund space is rotund.

In his original paper [39] Lovaglia gave the following example of a rotund space which is not locally uniformly rotund: Let C be the space of real-valued continuous functions defined on the interval [0,1]. Let $||f|| := \max|f(t)|$. Then if (t_n) is a dense sequence in [0,1] not including 0, C under the norm

$$||f||_1 := (||f|| + \sum_{n=1}^{\infty} \frac{1}{2^{2n}}|f(t_n)|^2)^{1/2}$$

is rotund but not locally uniformly rotund.

A useful characterization of local uniform rotundity in terms of sequences is the following (compare with proposition 2.1.3):

PROPOSITION 2.3.2: X is locally uniformly rotund if and only if for $x \in S$ and a sequence (x_n) in S, $||x_n+x|| \to 2$ implies $||x_n-x|| \to 0$.

Proof. Suppose X is locally uniformly rotund and $x \in S$, $(x_n) \subset S$ are such that $||x_n-x|| \geq \varepsilon$ for every $n \in \mathbb{N}$ for some $\varepsilon > 0$. Then there is a $\delta > 0$ such that $||x_n+x|| \leq 2 - \delta$ for every $n \in \mathbb{N}$, and so $||x_n+x|| \not\to 2$.

Conversely, if X is not locally uniformly rotund, then there exists $\varepsilon > 0$ and $x \in S$ such that for every $n \in \mathbb{N}^x$, there exists $x_n \in S$ with $||x_n-x|| \geq \varepsilon$ but $||x_n+x|| \geq 2 - \frac{1}{n}$. Then $||x_n+x|| \to 2$ but $||x_n-x|| \not\to 0$.

Of course, this proposition is just the analytic version of the geometric interpretation given earlier. However, it also suggests the following generalization.

DEFINITION (Lindenstrauss [36]): X is said to be weakly locally uniformly rotund whenever for $x \in S$ and a sequence (x_n) in S, $||x_n+x|| \to 2$ implies that (x_n) converges weakly to x.

Clearly a locally uniformly rotund space is weakly locally uniformly rotund, and a weakly locally uniformly rotund space is rotund. We have seen that l_∞ admits an equivalent rotund norm. However, Lindenstrauss [36] has shown that l_∞ cannot be equivalently renormed so as to be weakly locally uniformly rotund.

PROPOSITION 2.3.3: If X is weakly locally uniformly rotund, then X is very rotund.

Proof. Suppose that X is not very rotund. Then there exists $x \in S$, $x^* \in S^*$, and $x^{**} \in S^{**}$ such that

$$Q_0(x)(x^*) = 1 = x^{**}(x^*)$$

$$Q_0(x) \neq x^{**}$$

Let $y^* \in S^*$ be chosen so that

$$x^{**}(y^*) - Q_0(x)(y^*) =: \varepsilon > 0.$$

By the principle of local reflexivity, there exists for each $n \in \mathbb{N}^\times$ a linear map T_n:span$\{x,x^{**}\} \to X$ such that

(a) $T_n(x) = x$

(b) $z^*(T_n(x^{**})) = x^{**}(z^*)$ for $z^* \in \{x^*,y^*\}$

(c) $1 - \frac{1}{n} \leq ||T_n(x^{**})|| \leq 1 + \frac{1}{n}$.

Put

$$x_n := \frac{T_n(x^{**})}{||T_n(x^{**})||} \quad (n \in \mathbb{N}^\times)$$

Then (x_n) is a sequence in S. Moreover,

$$2 \geq ||x_n+x|| \geq x^*(x_n+x) = \frac{x^{**}(x^*)}{||T_n(x^{**})||} + x^*(x)$$

$$= \frac{1}{||T_n(x^{**})||} + 1 \geq \frac{n}{n+1} + 1 .$$

Hence $||x_n+x|| \to 2$. However,

$$y^*(x_n) - y^*(x) = \frac{x^{**}(y^*)}{||T_n(x^{**})||} - y^*(x)$$

whence $y^*(x_n) - y^*(x) \to \varepsilon > 0$. Thus X is not weakly locally uniformly rotund. ▨

If we now wish to consider weak* convergence, then we must step up to X*.

DEFINITION: X* is said to weak* locally uniformly rotund whenever for $x^* \in S^*$ and a sequence (x_n^*) in S*, $||x_n^* + x^*|| \to 2$ implies that x_n^* converges weak* to x*.

PROPOSITION 2.3.4: If X* is weak* locally uniformly rotund, then X* is rotund.

Proof. If $||x^* + y^*|| = 2$ for $x^*, y^* \in S^*$, then $x^* - y^*$ "converges" weak* to 0 in X, i.e. $x^* = y^*$. ▨

COROLLARY 2.3.5: If X** is weak* locally uniformly rotund, then X is very rotund.

The partial duality exhibited in propositions 2.1.6 and 2.2.2 fails slightly for the locally uniform properties.

PROPOSITION 2.3.6

(i) If X* is locally uniformly rotund, then X is locally uniform smooth.

(ii) If X* is locally uniformly smooth and X is weakly locally uniformly rotund, then X is locally uniformly rotund.

Proof. (i) Suppose X* is locally uniformly rotund. Let $x \mapsto f_x$ be a support mapping. If $\varepsilon > 0$ and $x \in S$, then there exists $\delta(\varepsilon, x) > 0$ such that if $y \in S$ and $||f_x + f_y|| > 2 - \delta$, then $||f_x - f_y|| < \varepsilon$.

Now suppose for $y \in S$ we have $||x-y|| < \delta$. Then

$$\begin{aligned}2 &= f_x(x) + f_y(y)\\ &= (f_x + f_y)(x) + f_y(y-x)\\ &\leq ||f_x + f_y|| + ||x-y||.\end{aligned}$$

Therefore $||f_x + f_y|| \geq 2 - ||x-y|| > 2 - \delta$, and so $||f_x - f_y|| < \varepsilon$. Every support mapping is thus norm-to-norm continuous on S, so X is locally uniformly smooth.

(ii) Let $x \in S$ and suppose (x_n) is a sequence in S such that $||x_n+x|| \to 2$. Let $x^* \in S^*$ be chosen so that $x^*(x) = 1$. Since X is weakly locally uniformly rotund, $x^*(x_n) \to x^*(x) = 1$. Then by Bollobás' extension of the Bishop-Phelps theorem, there exists sequences (y_n) in S and (y_n^*) in S* such that $y_n^*(y_n) = 1$ and

(14) $$||y_n^* - x^*|| \to 0$$

(15) $$||y_n - x_n|| \to 0$$

Since X* is locally uniformly smooth, the unique support mapping from S* to S** is norm-to-norm continuous. By (14) we must therefore have $||Q_0(y_n)-Q_0(x)|| \to 0$, i.e. $||y_n-x|| \to 0$. This result, coupled with (15), now shows that $||x_n-x|| \to 0$, and so X is locally uniformly rotund.

It would be nice if we could remove the additional assumption in proposition 2.3.6(ii), but unfortunately I do not know how this can be done.

2.4 Uniform smoothness and uniform rotundity

We now remove the "local" requirements from the results of the previous section, and consider (11) when δ may be chosen independently of both x and y.

DEFINITION: X is said to be uniformly smooth whenever given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in S$

and $||y|| < \delta$, then

$$||x+y|| + ||x-y|| < 2 + \varepsilon||y||$$

(equivalently, if $x,y \in S$, then (11) holds.)

DEFINITION (Clarkson [7]): X is said to be uniformly rotund whenever given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x,y \in S$ and $||x-y|| \geq \varepsilon$, then $||x+y|| \leq 2 - \delta$.

If the midpoint of a variable chord in the unit sphere of a uniformly rotund space thus approaches the boundary of the sphere, the length of the chord approaches zero. Lovaglia has given an example [39] of a Banach space which is locally uniformly rotund but not uniformly rotund.

PROPOSITION 2.4.1: The following are equivalent:

(i) X is uniformly smooth;

(ii) X^* is uniformly rotund;

(iii) Every support mapping is norm-to-norm uniformly continuous from S to S^*;

(iv) There exists a support mapping which is norm-to-norm uniformly continuous from S to S^*;

(v) X is uniformly Fréchet differentiable.

Proof. (i) implies (ii): Suppose X is uniformly smooth. Let $\varepsilon > 0$. Then there exists $\delta(\varepsilon) > 0$ such that if $x \in S$ and $||y|| < \delta$ we have

$$||x+y|| + ||x-y|| < 2 + \frac{\varepsilon||y||}{2} .$$

Suppose $x^*,y^* \in S^*$ are such that $||x^* - y^*|| \geq \varepsilon$. Then for some $y \in X$ with $||y|| = \delta/2$, we have

$$|(x^* - y^*)(y)| \geq \frac{\varepsilon\delta}{2}$$

and so

$$\begin{aligned} \|x^* + y^*\| &= \sup\{(x^* + y^*)(x) : x \in S\} \\ &= \sup\{x^*(x+y)+y^*(x-y)-(x^* + y^*)(y) : x \in S\} \\ &\leq \sup\{\|x+y\| + \|x-y\| - \frac{\varepsilon\delta}{2} : x \in S\} \\ &\leq 2 + \frac{\varepsilon\|y\|}{2} - \frac{\varepsilon\delta}{2} \\ &= 2 + \frac{\varepsilon\delta}{4} - \frac{\varepsilon\delta}{2} \\ &= 2 - \frac{\varepsilon\delta}{4}. \end{aligned}$$

Hence X^* is uniformly rotund.

(ii) implies (iii): Let $x \mapsto f_x$ be a support mapping. Let $\varepsilon > 0$. Then there exists $\delta(\varepsilon) > 0$ such that for $x,y \in S$, if $\|f_x + f_y\| \geq 2 - \delta$ then $\|f_x - f_y\| \leq \varepsilon$. So suppose $x,y \in S$ and $\|x-y\| < \delta$. Then as we have seen before,

$$2 = f_x(x) + f_y(y) \leq \|f_x + f_y\| + \|y-x\|,$$

so that $\|f_x + f_y\| \geq 2 - \delta$. Thus $\|f_x - f_y\| \leq \varepsilon$, whence we conclude that the support mapping is norm-to-norm uniformly continuous on S.

The proof that (iii) implies (iv) implies (v) implies (i) follows exactly the format in the proof of proposition 2.3.1 with the obvious modifications. ▨

The following sequential characterization of uniform rotundity corresponds to proposition 2.3.2 and is proved similarly.

<u>PROPOSITION 2.4.2</u>: X is uniformly rotund if and only if for sequences (x_n) and (y_n) in S, $\|x_n+y_n\| \to 2$ implies $\|x_n-y_n\| \to 0$.

As before, we may now weaken this proposition by asking for weak and weak* convergence rather than norm convergence.

<u>DEFINITION</u>: X is said to be <u>weakly uniformly rotund</u> whenever for sequences (x_n) and (y_n) in S, $||x_n+y_n|| \to 2$ implies that x_n-y_n converges weakly to zero.

<u>DEFINITION</u>: X* is said to be <u>weak* uniformly rotund</u> whenever for sequences (x_n^*) and (y_n^*) in S, if $||x_n^* + y_n^*|| \to 2$ then $x_n^* - y_n^*$ converges weak* to zero.

It is immediate that the uniform properties imply the corresponding locally uniform properties.

Although weak* uniform rotundity can not in general be defined on X, we shall nevertheless show that it is equivalent to a property on X.

<u>PROPOSITION 2.4.3</u>: The following are equivalent:

(i) X* is weak* uniformly rotund;

(ii) For every support mapping $x \mapsto f_x$, the real-valued mapping $x \mapsto f_x(y)$ is uniformly continuous on S for each $y \in S$;

(iii) There exists a support mapping $x \mapsto f_x$ such that the real-valued mapping $x \mapsto f_x(y)$ is uniformly continuous on S for each $y \in S$;

(iv) X is uniformly Gateaux differentiable.

Proof. (i) implies (ii): Suppose $x \mapsto f_x$ is a support mapping for which there exists a $y \in S$ such that the map $x \mapsto f_x(y)$ is not uniformly continuous on S. Then there exist sequences (x_n) and (z_n) in S such that $||x_n-z_n|| \to 0$ but

(16) $$|x_n^*(y) - z_n^*(y)| > \varepsilon > 0 \quad (n \in N)$$

where $x_n^* := f_{x_n}$ and $z_n^* := f_{z_n}$. As we have seen before,

$$2 \leq ||x_n^* + z_n^*|| + ||x_n-z_n||,$$

so we must have $||x_n^* + z_n^*|| \to 2$. By (16), however, $x_n^* - y_n^*$ does not converge weak* to zero. Thus X* is not weak* uniformly rotund.

(ii) implies (iii) is obvious.

(iii) implies (iv) follows from inequality (1).

(iv) implies (i): Suppose X is uniformly Gateaux differentiable. Let (x_k^*) and (y_k^*) be sequences in S* with $||x_k^* + y_k^*|| \to 2$. If $y \in S$ and $\varepsilon > 0$, then there exists a $\beta(\varepsilon,y) > 0$ such that

$$\frac{p(x,\beta y)}{\beta} \leq \frac{\varepsilon}{2} \quad (x \in S).$$

Then

$$\frac{||x+\beta y|| + ||x-\beta y||}{\beta} \leq \frac{\varepsilon}{2} + \frac{2}{\beta} \quad (x \in S). \tag{17}$$

Also, there exists $n_o \in \mathbb{N}$ such that

$$2 - \frac{\varepsilon}{2\beta} < ||x_k^* + y_k^*|| \quad (k > n_o).$$

For each $k > n_o$ choose $x_k \in S$ such that

$$2 - \frac{\varepsilon}{2\beta} \leq (x_k^* + y_k^*)(x_k).$$

Then for $k > n_o$

$$\begin{aligned} x_k^*(y)-y_k^*(y) &= \tfrac{1}{\beta}(x_k^*(\beta y+x_k)+y_k^*(x_k-\beta y)-(x_k^*(x_k)+y_k^*(x_k))) \\ &\leq \frac{||x_k+\beta y|| + ||x_k-\beta y||}{\beta} - \frac{x_k^*(x_k) + y_k^*(x_k)}{\beta} \\ &\leq \frac{\varepsilon}{2} + \frac{2 - (x_k^*(x_k) + y_k^*(x_k))}{\beta} \\ &\leq \varepsilon \end{aligned}$$

But $p(x,\beta y) = p(x,\beta(-y))$, so the above argument also shows that $x_k^*(-y)-y_k^*(-y) \leq \varepsilon$ for $k > n_o$. Hence

$$|x_k^*(y) - y_k^*(y)| \leq \varepsilon \quad (k > n_o)$$

and so $x_k^*(y) - y_k^*(y) \to 0$. Since y was arbitrary in S, we conclude that $x_k^* - y_k^*$ converges weak* to zero. Therefore X^* is weak* uniformly rotund. ▨

We shall see later that every uniformly rotund space is reflexive. Assuming this result for the moment, we can establish the complete duality between uniform smoothness and uniform rotundity.

PROPOSITION 2.4.4

(i) X^* is uniformly rotund if and only if X is uniformly smooth.

(ii) X^* is uniformly smooth if and only if X is uniformly rotund.

Proof. (i) This is part of proposition 2.4.1.

(ii) If X^* is uniformly smooth, then X^{**} is uniformly rotund by (i), and so X is uniformly rotund. Conversely, if X is uniformly rotund, then X is reflexive. Thus X^{**} is uniformly rotund and therefore by (i) X^* is uniformly smooth. ▨

2.5 Extreme smoothness and extreme rotundity

If X^{**} is smooth, there exists a unique support functional for each point in S^{**}. In this section we remove the uniqueness requirement, but ask instead that the support functionals for a fixed point in S^{**} all agree on X.

DEFINITION (Sullivan [57]): X is said to be extremely smooth if whenever

$$x^{***}(x^{**}) = 1 = y^{***}(x^{**})$$

for $x^{***}, y^{***} \in S^{***}$ and $x^{**} \in S^{**}$, then

$$x^{***} - y^{***} \in X^{\perp}.$$

Clearly if X^{**} is smooth then X is extremely smooth.

<u>PROPOSITION 2.5.1</u>: If X is extremely smooth, then X^* is rotund.

Proof. Suppose $x^{**} \in S^{**}$ and

$$x^{**}(x^*) = 1 = x^{**}(y^*)$$

for $x^*, y^* \in S^*$. Then

$$Q_1(x^*)(x^{**}) = 1 = Q_1(y^*)(x^{**})$$

and so $Q_1(x^*) - Q_1(y^*) \in X^{\perp}$, i.e. $x^* = y^*$. ▨

<u>COROLLARY 2.5.2</u>

(i) If X is extremely smooth, then X is smooth.

(ii) If X^* is extremely smooth, the X is very rotund.

The space l_1 admits an equivalent norm under which it is smooth [14], but by proposition 2.5.1 it can not then be extremely smooth since $(l_1)^*$ has no equivalent rotund norm [14].

The next proposition characterizes extreme smoothness by use of sequences.

<u>PROPOSITION 2.5.3</u>: X is extremely smooth if and only if whenever (x_k^*) and (y_k^*) are sequences in S^* such that

$$\lim_k x^{**}(x_k^*) = 1 = \lim_k x^{**}(y_k^*) \tag{18}$$

for some $x^{**} \in S^{**}$, then $x_k^* - y_k^*$ converges weak* to zero.

Proof. Suppose (x_k^*) and (y_k^*) are sequences in S^* satisfying (18) for some $x^{**} \in S^{**}$, while $x_k^*(y) - y_k^*(y) > \varepsilon > 0$ for some $y \in S$ and for all $k \in \mathbb{N}$. If x^{***} and y^{***} are weak* limit points in B^{***} of (x_k^*) and (y_k^*) respectively, then by (18) x^{***} and y^{***} belong to S^{***} and $x^{***}(x^{**}) = 1 = y^{***}(x^{**})$. However, $(x^{***} - y^{***})(y) > \varepsilon > 0$, and so X is not extremely smooth.

Conversely, suppose X is not extremely smooth. Then there exists $x^{***}, y^{***} \in S^{***}$, $x^{**} \in S^{**}$, and $x \in S$ such that

$$x^{***}(x^{**}) = 1 = y^{***}(x^{**})$$

but

$$x^{***}(x) - y^{***}(x) =: \varepsilon > 0.$$

By the principle of local reflexivity we can find for each $n \in \mathbb{N}^{\times}$ a linear map $T_n : \operatorname{span}\{x^{***}, y^{***}\} \to X^*$ such that

$$\text{(a) } y^{**}(T_n(z^{***})) = z^{***}(y^{**})$$
$$\text{(b) } 1 - \frac{1}{n} \leq ||T_n(z^{***})|| \leq 1 + \frac{1}{n}$$

for $y^{**} \in \{x, x^{**}\}$ and $z^{***} \in \{x^{***}, y^{***}\}$. Put

$$x_n^* := \frac{T_n(x^{***})}{||T_n(x^{***})||}$$
$$y_n^* := \frac{T_n(y^{***})}{||T_n(y^{***})||}$$

for each $n \in \mathbb{N}^{\times}$. Then

$$\lim_n x^{**}(x_n^*) = 1 = \lim_n x^{**}(y_n^*)$$

but $(x_n^*(x) - y_n^*(x)) \to (x^{***}(x) - y^{***}(x)) = \varepsilon > 0.$ ▨

We have seen that if X^{**} is smooth then X is both very smooth and extremely smooth. Smith showed [51] that if $\dim(X^{**}/X) \leq 1$ then the converse also holds.

<u>PROPOSITION 2.5.4</u>: If $\dim(X^{**}/X) \leq 1$, then X^{**} is

smooth if and only if X is very smooth and extremely smooth.

Proof. Suppose X is both very smooth and extremly smooth. If $\dim(X^{**}/X) \leq 1$, then

$$X^{**} = X + \operatorname{span}\{b^{**}\}$$

for some $b^{**} \in X^{**}$. Suppose $x^{**} \in S^{**}$ and $x^{***}, y^{***} \in S^{***}$ satisfy

$$x^{***}(x^{**}) = 1 = y^{***}(x^{**}).$$

Write $x^{**} = x + \beta b^{**}$ for $x \in X$ and $\beta \in \mathbb{R}$. Since X is very smooth, $\beta = 0$ implies $x^{***} = y^{***}$. So now suppose $\beta \neq 0$. Since X is extremely smooth, $x^{***} - y^{***} \in X^{\perp}$, and so

$$0 = (x^{***} - y^{***})(x^{**}) = (x^{***} - y^{***})(\beta b^{**}).$$

Since $\beta \neq 0$, we must then have $(x^{***} - y^{***})(b^{**}) = 0$. Hence $(x^{***} - y^{***})(X^{**}) = 0$, i.e. $x^{***} = y^{***}$.

X^{**} is therefore smooth. ▨

In the theory of best approximation in normed spaces, Chebychev sets play an important role. These are sets such that each element of the space has a unique best approximation in the set, i.e. M is a Chebychev set in Y if for each $y \in Y$ there exists a unique element $p(y) \in M$ such that

$$||y-x|| > ||y-p(y)|| \quad (x \in M \setminus \{p(y)\}).$$

We shall see how this property is related to the rotundity property.

DEFINITION (Sullivan [57]): X is said to be *extremely rotund* if every finite dimensional subspace of X is a Chebychev subspace of X**.

PROPOSITION 2.5.5: If X is extremely rotund, then X is rotund.

Proof. Suppose X is extremely rotund, and suppose $x,y \in S$ are such that

$$||(1-\beta)x + \beta y|| = 1 \quad (0 \leq \beta \leq 1).$$

Then for $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$

$$\left|\left| \frac{x+y}{2} - \alpha(x-y)\right|\right| = \left|\left|\left(\frac{1}{2} - \alpha\right)x + \left(\frac{1}{2} + \alpha\right)y\right|\right| = 1,$$

while for $|\alpha| > \frac{1}{2}$

$$\left|\left| \frac{x+y}{2} - \alpha(x-y)\right|\right| \geq \left| \left|\frac{1}{2} - \alpha\right| - \left|\frac{1}{2} + \alpha\right| \right| = 1.$$

Thus each $\alpha(x-y)$ for $|\alpha| \leq \frac{1}{2}$ is a best approximation to $\frac{x+y}{2}$ in the subspace spanned by x-y in X**. Since X is extremely rotund, we must therefore have x = y. Hence X is rotund.

2.6 Strong extreme smoothness and strong extreme rotundity

DEFINITION (Sullivan [57]): X is said to be strongly extremely smooth if in X***,

$$||x^* + x^\perp|| = 1 = ||y^* + y^\perp||$$
$$||(x^* + x^\perp) + (y^* + y^\perp)|| = 2$$

imply that $x^* = y^*$.

The following result indicates the relative strength of this property.

PROPOSITION 2.6.1

(i) If X is strongly extremely smooth, then X is extremely smooth.

(ii) If X is uniformly Gateaux differentiable, then X is strongly extremely smooth.

Proof. (i) Suppose $||x^* + x^\perp|| = 1 = ||y^* + y^\perp||$ and $(x^* + x^\perp)(x^{**}) = 1 = (y^* + y^\perp)(x^{**})$ for some $x^{**} \in S^{**}$. Then $||(x^* + x^\perp) + (y^* + y^\perp)|| = 2$, so by the strong extreme smoothness of X, we have $x^* = y^*$. Therefore

$$(x^* + x^\perp) - (y^* + y^\perp) = x^\perp - y^\perp \in X^\perp .$$

(ii) Suppose $||x^* + x^\perp|| = 1 = ||y^* + y^\perp||$ and $||(x^* + x^\perp) + (y^* + y^\perp)|| = 2$, but $(x^* - y^*)(z) =: \varepsilon > 0$ for some $z \in S$. By the principle of local reflexivity, we obtain for each $n \in \mathbb{N}^\times$ a linear map $T_n : \mathrm{span}\{x^* + x, y^* + y\} \to X^*$ such that

(a) $Q_0(z)(T_n(z^{***})) = z^{***}(z)$

(b) $(1 - \frac{1}{n})||z^{***}|| \leq ||T_n(z^{***})|| \leq (1 + \frac{1}{n})||z^{***}||$

for $z^{***} \in \mathrm{span}\{x^* + x^\perp, y^* + y^\perp\}$. For each $n \in \mathbb{N}^\times$ put

$$x_n^* := \frac{T_n(x^* + x^\perp)}{||T_n(x^* + x^\perp)||}$$

$$y_n^* := \frac{T_n(y^* + y^\perp)}{||T_n(y^* + y^\perp)||}$$

Then for each $z^{**} \in S^{**}$

$$z^{**}(x_n^* + y_n^*) \geq \frac{n}{n+1} z^{**}(T_n(x^* + x^\perp + y^* + y^\perp)),$$

so that

$$2 \geq ||x_n^* + y_n^*||$$
$$\geq \frac{n}{n+1}||T_n(x^* + x^\perp + y^* + y^\perp)||$$
$$\geq 2\left(\frac{n-1}{n+1}\right).$$

Hence $||x_n^* + y_n^*|| \to 2$, but it is easily seen that

$$(x_n^* - y_n^*)(z) \to (x^* - y^*)(z) = \varepsilon > 0.$$

Since X* is therefore not weak* uniformly rotund, X is not uniformly Gateaux differentiable. ▨

Notice that if X*** is rotund then X is strongly extremely smooth, for if $||(x^* + x^\perp) + (y^* + y^\perp)|| = 2$ for norm-1 vectors, we have $x^* + x^\perp = y^* + y^\perp$.

If X** is rotund, then $||x^{**} + y^{**}|| = 2$ for norm-1 vectors implies that 0 equals $x^{**} - y^{**}$. The next rotundity property says that if $||x^{**} + y^{**}|| = 2$, then 0 is a best approximation to $x^{**} - y^{**}$ in X (or equivalently, that $x^{**} - y^{**}$ is orthogonal to X in the Birkhoff sense.)

<u>DEFINITION</u> (Sullivan [57]): X is said to be <u>strongly extremely rotund</u> if in X**,

$$||x^{**}|| = 1 = ||y^{**}||$$
$$||x^{**} + y^{**}|| = 2$$

imply that $\text{dist}(x^{**} - y^{**}, X) = ||x^{**} - y^{**}||$.

Thus if X** is rotund, X is clearly strongly extremely rotund.

PROPOSITION 2.6.2: If X is strongly extremely rotund, then X is extremely rotund.

Proof. If X is not extremely rotund, there exists some finite-dimensional subspace of X which is not a Chebychev subspace in X**. By a theorem of Singer [48, p.105], this implies the existence of elements $x^{***} \in S^{***}$, $x^{**} \in S^{**}$, and $x \in X$, $x \neq 0$, such that $x^{***}(x) = 0$ and

$$x^{***}(x^{**}) = ||x^{**}|| = ||x^{**} - x|| = 1.$$

Then

$$2 \geq ||(x^{**} - x) + x^{**}|| \geq x^{***}((x^{**} - x) + x^{**}) = 2.$$

Thus $||(x^{**} - x) + x^{**}|| = 2$ but

$$||(x^{**} - x) - x^{**}|| = ||x|| \neq 0.$$

Since $\text{dist}((x^{**} - x) - x^{**}) = 0$, X is not strongly extremely rotund. ▨

COROLLARY 2.6.3: If X* is extremely smooth, then X is extremely rotund.

Recall from the introduction that $P_1 := Q_1 Q_0{}^*$ is the norm-1 projection on X*** with range X* and null space $X^{\perp}$. Let $P_2 := Q_2 Q_1{}^*$ denote the corresponding projection on $X^{(4)}$ with range X** and null space $X^{*\perp}$. Brown has shown [6] that $||Q_2 - Q_0{}^{**}|| = ||I - P_1||$. This follows from the observation that

$$((Q_2 - Q_0{}^{**})(x^{**}))(y^{***}) = ((I - P_1)(y^{***}))(x^{**})$$

for all $x^{**} \in X^{**}$ and $y^{***} \in X^{***}$.

Since $||P_1|| = 1$, we have $||I - P_1|| \leq 2$. If X is reflexive, then $||I - P_1|| = 0$. If X is not reflexive, there exists $x^\perp \in X^\perp$ with $||(I - P_1)(x^\perp)|| = ||x^\perp|| = 1$, so that $||I - P_1|| \geq 1$. For $X = c_0$, $||I - P_1|| = 1$, while Brown also showed in [6] that for $X = l_1$, $||I - P_1|| = 2$.

<u>LEMMA 2.6.4</u>: $||I - P_1|| \leq 1$ if and only if $\text{dist}(x^\perp, X^*) = ||x^\perp||$ for all $x^\perp \in X^\perp$.

Proof. $||x^\perp|| = \text{dist}(x^\perp, X^*)$ for all $x^\perp \in X^\perp$ if and only if $||(I - P_1)(x^\perp + x^*)|| = ||x^\perp|| \leq ||x^\perp + x^*||$ for all $x^* \in X^*$ and all $x^\perp \in X^\perp$ if and only if $||I - P_1|| \leq 1$. ▨

Compare this with the result that for any Banach space $||x^*|| = \text{dist}(x^*, X^\perp)$.

<u>PROPOSITION 2.6.5</u>: If $||I - P_1|| \leq 1$, then X is strongly extremely smooth if and only if X* is strongly extremely rotund.

Proof. If X is strongly extremely smooth, and if $||x^{***} + y^{***}|| = 2$ for $x^{***}, y^{***} \in S^{***}$, we have $x^{***} - y^{***} \in X^\perp$. Hence by the lemma, $||x^{***} - y^{***}|| = \text{dist}(x^{***} - y^{***}, X^*)$, and so X* is strongly extremely rotund.

Conversely, suppose $||x^* + x^\perp|| = 1 = ||y^* + y^\perp||$ and $||(x^* + x^\perp) + (y^* + y^\perp)|| = 2$. Then

$$\begin{aligned} ||x^\perp - y^\perp|| &\geq \text{dist}((x^\perp - y^\perp) + (x^* - y^*), X^*) \\ &= ||(x^\perp - y^\perp) + (x^* - y^*)|| \\ &= ||x^\perp - y^\perp||, \end{aligned}$$

where the first equality follows from X* being strongly extremely rotund and the second from $||I - P_1|| \leq 1$. Thus

$$\begin{aligned} ||(x^{\perp} - y^{\perp}) + (x^* - y^*)|| &= ||x^{\perp} - y^{\perp}|| \\ &\geq \text{dist}(x^{\perp} - y^{\perp}, M) \\ &\geq \text{dist}(x^{\perp} - y^{\perp}, X^*) \\ &= ||x^{\perp} - y^{\perp}||, \end{aligned}$$

where M is the subspace spanned by $x^* - y^*$ in X^{***}. Since 0 and $-(x^* - y^*)$ are therefore both best approximations to $x^{\perp} - y^{\perp}$ in M, we must have $x^* - y^* = 0$, i.e. $x^* = y^*$. Hence X is strongly extremely smooth. ▨

2.7 Hahn-Banach smoothness and the Namioka-Phelps property

Let $x^* \in X^*$. If we consider X as a subspace of X^{**}, the $Q_1(x^*)$ is a Hahn-Banach extension of $x^*|_X$, i.e. $||Q_1(x^*)|| = ||x^*||$ and $Q_1(x^*)|_X = x^*|_X$. Of course, there is no apparent reason why $x^*|_X$ might not have another norm-1 extension $x^* + x^{\perp}$ in X^{***}.

DEFINITION (Sullivan [57]): X is said to be Hahn-Banach smooth if in X^{***}, $||x^* + x^{\perp}|| = 1 = ||x^*||$ implies that $x^{\perp} = 0$.

Thus X is Hahn-Banach smooth if and only if $Q_1(x^*)$ is the unique Hahn-Banach extension of $x^*|_X$ for each $x^* \in S^*$. According to Sullivan [57], Hahn-Banach smooth implies neither smooth nor very smooth.

Namioka and Phelps introduced a property (**) in their paper [41] in which weak* and norm convergence of nets coincide on the unit sphere of X^*:

DEFINITION: X^* is said to have the Namioka-Phelps property if for every net (x_α^*) in X^*,

$$x_\alpha^* \xrightarrow{w^*} x^*$$

$$||x_\alpha^*|| \to ||x^*||$$

imply $||x_\alpha^* - x^*|| \to 0$.

<u>PROPOSITION 2.7.1</u>: If X^* has the Namioka-Phelps property, then X is Hahn-Banach smooth.

Proof. Suppose that in X^{***} we have

$$||x^*|| = 1 = ||x^* + x^\perp||.$$

Let E be the two-dimensional subspace of X^{***} spanned by x^* and $x^\perp$. Let Ω denote the collection of non-zero finite-dimensional subspaces of X directed by inclusion. From the principle of local reflexivity, we have for each $\alpha \in \Omega$ a linear map $T_\alpha : E \to X^*$ such that

(a) $T_\alpha(x^*) = x^*$

(b) $Q_0(x_\alpha)(T_\alpha(x^* + x^\perp)) = (x^* + x^\perp)(x_\alpha) \quad (x_\alpha \in \alpha)$

(c) for each $e \in E$

$$(1 - \frac{1}{\dim(\alpha)})||e|| \leq ||T_\alpha(e)|| \leq (1 + \frac{1}{\dim(\alpha)})||e||$$

Put $x_\alpha^* := T_\alpha(x^* + x^\perp)$. Then (b) can be replaced by

(b') $x_\alpha^*(x_\alpha) = x^*(x_\alpha)$ for each $x_\alpha \in \alpha$.

Let $y \in X$ and let α be the subspace of X spanned by y. Then for every $\beta \geq \alpha$, (b') says that $x_\beta^*(y) = x^*(y)$, and so $x_\beta^*(y) \to x^*(y)$. Since y was arbitrary, we conclude that (x_β^*) converges weak* to x^* in X^*.

Now (c) shows that $||T_\alpha(e)|| \to ||e||$ for each $e \in E$.

In particular,

$$||x_\alpha^*|| = ||T_\alpha(x^* + x^\perp)|| \to ||x^* + x^\perp|| = ||x^*||.$$

Therefore $||x_\alpha^* - x^*|| \to 0$ if X^* has the Namioka-Phelps property. But $x_\alpha^* - x^* = T_\alpha(x^\perp)$, so again by (c) we have $||x^\perp|| = 0$, i.e. $x^\perp = 0$. Thus X is Hahn-Banach smooth. ▨

PROPOSITION 2.7.2: If X^{***} is rotund, then X is Hahn-Banach smooth.

Proof. Suppose $||x^* + x^\perp|| = 1 = ||x^*||$. Let (x_n) be a sequence in S such that $x^*(x_n) \to 1$. Then

$$((x^* + x^\perp) + x^*)(x_n) \to 2$$

and so $||(x^* + x^\perp) + x^*|| = 2$. Since X^{***} is rotund, we have $x^* = x^* + x^\perp$, i.e. $x^\perp = 0$.

If X^{***} is rotund, then, X is both strongly extremely smooth and Hahn-Banach smooth. In [54] it is shown that if $\dim(X^{**}/X) \leq 1$, then the converse also holds (compare with proposition 2.5.4.)

PROPOSITION 2.7.3: If $\dim(X^{**}/X) \leq 1$, then X^{***} is rotund if and only if X is both strongly extremely smooth and Hahn-Banach smooth.

Proof. Since $X^\perp \cong (X^{**}/X)^*$, we have $\dim(X^\perp) \leq 1$. Therefore $X^{***} = X^* \oplus \text{span}\{x^\perp\}$ for some $x^\perp \in X^\perp$.

Now suppose $||x^* + \alpha x^\perp|| = 1 = ||y^* + \beta x^\perp||$ and $||(x^* + \alpha x^\perp) + (y^* + \beta x^\perp)|| = 2$. Since X is strongly extremely smooth, $x^* = y^*$. Choose $x^{(4)} \in S^{(4)}$ such that

$$x^{(4)}(x^* + \alpha x^\perp) = 1 = x^{(4)}(x^* + \beta x^\perp).$$

Then $(\alpha-\beta)x^{(4)}(x^{\perp}) = 0$. If $x^{(4)}(x^{\perp}) \neq 0$, we have $\alpha = \beta$, and so $x^* + \alpha x^{\perp} = y^* + \beta x^{\perp}$. On the other hand, if $x^{(4)}(x^{\perp}) = 0$, then $x^{(4)}(x^*) = 1$, and so $||x^*|| = 1$. Thus

$$||x^* + \alpha x^{\perp}|| = ||x^*|| = 1 = ||x^* + \beta x^{\perp}||.$$

Since X is Hahn-Banach smooth we must have $\alpha = 0 = \beta$, and again we conclude that $x^* + \alpha x^{\perp} = y^* + \beta x^{\perp}$.

Therefore X^{***} is rotund. ▨

For the next property we require that $Q_1(x^*)$ be the unique Hahn-Banach extension of $x^*|_X$ only for those $x^* \in S^*$ which attain their norm on S.

DEFINITION (Smith and Sullivan [54]): X is said to be weakly Hahn-Banach smooth if in X^{***}, $||x^* + x^{\perp}|| = 1 = ||x^*||$ and $x^*(x) = 1$ for some $x \in S$ imply that $x^{\perp} = 0$.

By James' theorem, Hahn-Banach smooth and weakly Hahn-Banach smooth coincide in reflexive spaces.

PROPOSITION 2.7.4: X is very smooth if and only if X is smooth and weakly Hahn-Banach smooth.

Proof. Suppose X is very smooth. We have seen that X is therefore smooth. If in X^{***} we have $||x^* + x^{\perp}||=1=||x^*||$ and $x^*(x) = 1$ for some $x \in S$, then both x^* and $x^* + x^{\perp}$ are support functionals of x in X^{***}, and so $x^* = x^* + x^{\perp}$, i.e. $x^{\perp} = 0$.

Conversely, let $x \in S$. Since X is smooth, there exists a unique $x^* \in S^*$ such that $x^*(x) = 1$. Now suppose $y^* + y^{\perp} \in S^{***}$ is a support functional for x in X^{***}. Then $y^*(x) = 1$, and since $||y^*|| \leq ||y^* + y^{\perp}|| = 1$, we must have $||y^*|| = 1$. Therefore $y^* = x^*$. Moreover, the two statements

$$||y^* + y^\perp|| = 1 = ||y^*||$$
$$y^*(x) = 1$$

imply by weak Hahn-Banach smoothness that $y^\perp = 0$. Thus x^* is the unique support functional of x in X^{***}, whence X is very smooth. ▨

2.8 Chart of implications for the geometrical properties

In this section we summarize the relationships between the different properties in a chart of implications modelled from a paper of F. Sullivan [57]. If P is a property, then P^k means that the k^{th} conjugate of X possesses P. We write simply P instead of P^0. An arrow indicates an implication. The additional assumptions $||I - P_1|| \leq 1$ and $\dim(X^{**}/X) \leq 1$ are indicated by a + and ++ respectively. We use the following abbreviations:

S - smooth
VS - very smooth
LUS - locally uniformly smooth
US - uniformly smooth
ES - extremely smooth
SES - strongly extremely smooth
HBS - Hahn-Banach smooth
WHBS - weakly Hahn-Banach smooth
R - rotund
VR - very rotund
LUR - locally uniformly rotund
wLUR - weakly locally uniformly rotund
w*LUR - weak* locally uniformly rotund

UR - uniformly rotund

wUR - weakly uniformly rotund

w*UR - weak* uniformly rotund

ER - extremely rotund

SER - strongly extremely rotund

GD - Gateaux differentiable

FD - Fréchet differentiable

UGD - uniformly Gateaux differentiable

UFD - uniformly Fréchet differentiable

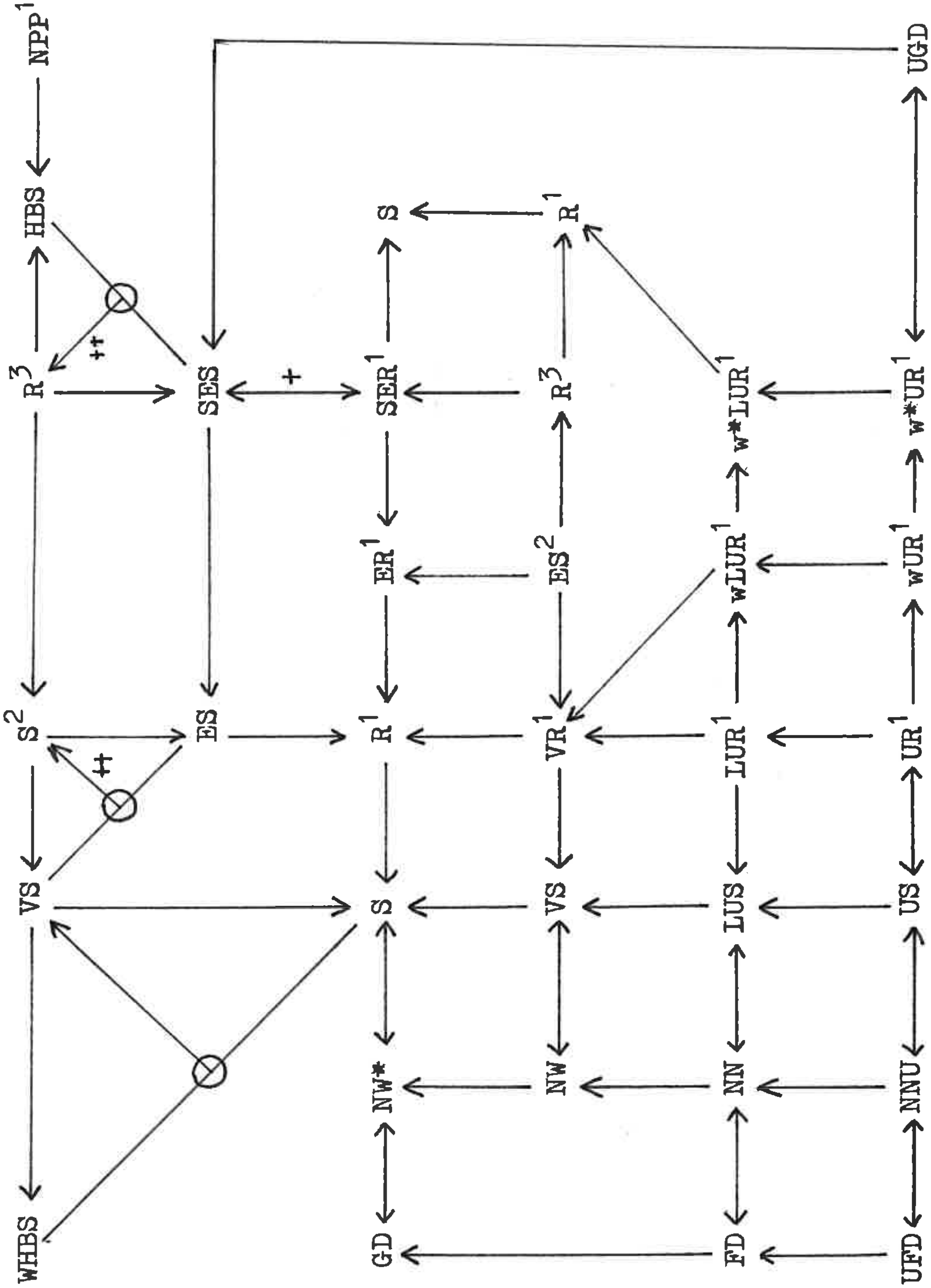
NPP - Namioka-Phelps property

NW* - norm-to-weak* continuous support mapping

NW - norm-to-weak continuous support mapping

NN - norm-to-norm continuous support mapping

NNU - norm-to-norm uniformly continuous support mapping


NPP¹
HBS
R³
SES
SER¹
S
R¹
R³
w*LUR¹
w*UR¹
UGD
S²
ES
ER¹
ES²
wLUR¹
wUR¹
VS
R¹
VR¹
LUR¹
UR¹
S
VS
LUS
US
NW*
NW
NN
NNU
WHBS
GD
FD
UFD
+
++
++

3. REFLEXIVITY

3.1 The Theorems

In this section we employ the properties and results from the previous chapter to determine sufficient conditions for a Banach space to be reflexive. The three main theorems we shall prove are that if either X is uniformly rotund, X* is very smooth, or X* is Hahn-Banach smooth, then X is reflexive. From these three (and in particular the second) numerous other sufficient conditions are derivable. Rather than list them as corollaries, we shall exhibit them in a chart of implications.

The first theorem we prove is due independently to D.P. Milman [40] and B.J. Pettis [43]. It is the previously promised result that every uniformly rotund space is reflexive. We shall give three proofs - the first uses James' theorem, the second Goldstine's theorem, and the third is an outline of Pettis' original proof [43].

THEOREM 3.1.1: If X is uniformly rotund, then X is reflexive.

Proof 1. Suppose X is uniformly rotund, and let $x^* \in S^*$. Choose a sequence (x_n) in S such that $x^*(x_n) \to 1$. We claim that (x_n) is Cauchy. If this is not true, then for some $\varepsilon > 0$ and some subsequence (y_n) of (x_n) we have

$$||y_k - y_j|| \geq \varepsilon \quad (k \neq j).$$

Then by uniform rotundity there exists $\delta(\varepsilon) > 0$ such that

$$||y_k + y_j|| \leq 2 - \delta \quad (k \neq j)$$

and so

$$x^*(y_k) + x^*(y_j) \leq 2 - \delta \quad (k \neq j)$$

Since this contradicts $x^*(y_n) \to 1$, we must indeed have (x_n) Cauchy. Therefore (x_n) converges to some $x \in S$ and $x^*(x) = 1$. By James' theorem X is consequently reflexive. ▨

Proof 2 (J. Ringrose). Suppose X is not reflexive but is uniformly rotund. Then there exists $x^{**} \in S^{**}$ such that $\mathrm{dist}(x^{**}, B) =: 2\varepsilon > 0$. It follows from uniform rotundity that there exists $\delta(\varepsilon) > 0$ such that for $x, y \in S$, $||x+y|| \geq 2-\delta$ implies $||x-y|| \leq \varepsilon$. Now choose $x^* \in S^*$ such that

$$|x^{**}(x^*) - 1| < \frac{\delta}{4} .$$

Define a weak* neighborhood V of x^{**} by

$$V := \{y^{**} \in X^{**} : |y^{**}(x^*) - 1| < \frac{\delta}{2}\}.$$

Suppose $x, y \in B \cap V$. Then

$$x^*(x+y) = (x^*(x) - 1) + (x^*(y) - 1) + 2 \geq 2 - \delta.$$

Thus $||x+y|| \geq 2 - \delta$, and so $||x-y|| \leq \varepsilon$.

Now fix $x \in B \cap V$. The above result then shows that

$$V \cap B \subset \{x\} + \varepsilon B^{**}.$$

By Goldstine's theorem, $V \cap B$ is weak* dense in $V \cap B^{**}$. Since $\{x\} + \varepsilon B^{**}$ is weak* closed, we obtain

$$V \cap B^{**} \subset \{x\} + \varepsilon B^{**}.$$

But $x^{**} \in V \cap B^{**}$, so $x^{**} = x + \varepsilon y^{**}$ for some $y^{**} \in B^{**}$. then

$$2\varepsilon = \text{dist}(x^{**}, B) \leq ||x^{**} - x|| = \varepsilon.$$

This contradiction shows that X must be reflexive. ▨

Proof 3 (B.J. Pettis). As we saw in proof 1, if X is uniformly rotund, then for each $f \in S^*$ there exists a unique $x \in S$ such that $f(x) = 1$. By James' theorem this sufficed to prove the theorem. But Pettis did not have the use of James' theorem! So let $F \in B^{**}$ and for each $n \in \mathbb{N}^x$ choose $f_n \in S^*$ such that $1 - \frac{1}{n} \leq F(f_n)$. Then there exists a unique sequence (x_n) in S such that $f_n(x_n) = 1$ for each n. Pettis proved that (x_n) is a Cauchy sequence whose limit x_o satisfies $F(f)=f(x_o)$ for every $f \in X^*$. To do this he used the observation that for any $F \in X^{**}$ there exists a non-negative real-valued bounded finitely additive measure γ defined on subsets of B such that

$$F(f) = \int_B f \, d\gamma \quad (f \in X^*)$$

and $||F|| = \text{variation of } \gamma = \gamma(B)$. ▨

THEOREM 3.1.2: If X is uniformly smooth, then X is reflexive.

Proof. From proposition 2.4.1, if X is uniformly smooth, then X^* is uniformly rotund, and hence X^* is reflexive. Therefore X is also reflexive. ▨

In 1941 V.L. Smulyan published a proof [55] that if X^* is Fréchet differentiable, then X is reflexive. Our next theorem is a stronger version of that result.

THEOREM 3.1.3: If X^* is very smooth, then X is reflexive.

Proof. Denote by $f \mapsto F_f$ the norm to weakly continuous support mapping of S^* into S^{**}. Let $f \in S^*$. We shall show that f attains its norm on S.

Since X is subreflexive, there exists sequences (f_n) in S^* and (x_n) in S such that

$$f_n(x_n) = 1 \tag{1}$$

$$||f-f_n|| \to 0 \tag{2}$$

Since X^* is smooth, we have from (1) that $x_n = F_{f_n}$. Since X^* is very smooth, we thus have from (2) that $x_n \to F_f$ weakly in X^{**}. Now X is norm closed in X^{**} and hence weakly closed in X^{**}. Thus $F_f \in X$, whence there exists an $x \in S$ satisfying

$$f(x) = F_f(f) = 1.$$

By James' theorem X is therefore reflexive. ▨

As immediate corollaries, we have that spaces with smooth third conjugate and spaces with rotund fourth conjugate are reflexive. The former result is due independently to J.R. Giles, M.M. Day and R.R. Phelps, while the latter was first proved by J. Dixmier [21]. Mark Smith has shown [50, 51] that the non-reflexive James' space [33] admits two equivalent norms with smooth second conjugate and rotund third conjugate respectively.

The next sufficient condition for reflexivity is due to Sullivan [57]:

<u>THEOREM 3.1.4</u>: If X^* is Hahn-Banach smooth, then X is reflexive.

Proof. First observe that for $x \in X$ and $x^* \in X^*$,

$$Q_0^* Q_1(x^*)(x) = Q_1(x^*)(Q_0(x)) = x^*(x),$$

i.e. $Q_1(x^*) \circ Q_0 = x^*$. Hence for $x^{**} \in X^{**}$,

$$Q_0^{**}(x^{**})(Q_1(x^*)) = x^{**}(Q_1(x^*) \circ Q_0) = x^{**}(x^*)$$

so that

(3) $$Q_0^{**}(x^{**}) - x^{**} \in X^{*\perp}.$$

Now suppose X is not reflexive. Then there exists an $x^{**} \in S^{**}$ with $x^{**} \notin X$. By the Hahn-Banach theorem, there exists $x^{\perp} \in X^{\perp}$ with $x^{\perp}(x^{**}) \neq 0$. Then

$$(x^{**} - Q_0^{**}(x^{**}))(x^{\perp}) = x^{\perp}(x^{**}) \neq 0$$

and so $x^{**} \neq Q_0^{**}(x^{**})$. Thus by (3) we can find $x^{*\perp} \in X^{*\perp}$ such that $x^{*\perp} \neq 0$ and $Q_0^{**}(x^{**}) = x^{**} + x^{*\perp}$. Therefore

$$||x^{**} + x^{*\perp}|| = ||Q_0^{**}(x^{**})|| = ||x^{**}|| = 1,$$

so that X^* is not Hahn-Banach smooth. ▨

Our last series of results establishes sufficient conditions for reflexivity in terms of extreme smoothness and strongly extreme rotundity. We first need the

<u>LEMMA 3.1.5</u>: If $||I - P_2|| \leq 1$, then $||I - P_1|| \leq 1$.

Proof. Suppose $||I - P_1|| > 1$. From Brown's theorem there exists an $x^{**} \in S^{**}$ such that

(4) $$||x^{**} - Q_0^{**}(x^{**})|| > 1.$$

Recall that $x^{**} - Q_0^{**}(x^{**}) \in X^{*\perp}$. Now

$$\text{dist}(x^{**} - Q_0^{**}(x^{**}), X^{**}) \leq ||Q_0^{**}(x^{**})|| = 1,$$

so that by (4) and lemma 2.6.4, we must have $||I - P_2|| > 1$. ▨

<u>THEOREM 3.1.6</u>: If $||I - P_2|| \leq 1$ and X^{**} is extremely smooth, then X is reflexive.

Proof. First we show that for any Banach space Y, if Y^* is extremely smooth and $||I - P_1|| \leq 1$, then

$$\text{dist}(y^{**}, Y) < 1 \quad (y^{**} \in S^{**}).$$

For if $\text{dist}(y^{**}, Y) = 1$ for some $y^{**} \in S^{**}$, then by the Hahn-Banach theorem there exists $y^\perp \in Y^\perp$ such that $||y^\perp|| = 1$ and $y^\perp(y^{**}) = 1$. Now $||I - P_1|| \leq 1$ implies that $\text{dist}(y^\perp, Y^*) = || y^\perp|| = 1$. Repeating the same argument, there exists $y^{*\perp} \in Y^{*\perp}$ such that $||y^{*\perp}|| = 1$ and $y^{*\perp}(y^\perp) = 1$. Then

$$y^{*\perp}(y^\perp) = 1 = Q_2(y^{**})(y^\perp).$$

Since X^* is extremely smooth, we have $y^{*\perp} - y^{**} \in Y^{*\perp}$, and hence $y^{**} \in Y^{*\perp} \cap Y^{**} = \{0\}$. This contradiction establishes our claim.

Thus X^{**} extremely smooth and $||I - P_2|| \leq 1$ imply that $\text{dist}(x^{***}, X^*) < 1$ if $||x^{***}|| = 1$. However, by the previous lemma, $||I - P_1|| \leq 1$, so that $\text{dist}(x^\perp, X^*) = 1$ if $||x^\perp|| = 1$. We must therefore have $X^\perp = \{0\}$! Hence $X^{***} = Q_1(X^*)$, i.e. X^* is reflexive, whence X is reflexive. ▨

Closer scrutiny of this theorem is necessary. If $||I - P_2|| \leq 1$, then either $||I - P_2|| = 0$, in which case X is clearly reflexive, or $||I - P_2|| = 1$, in which case X is not reflexive. Hence if $||I - P_2|| = 1$, X^{**} can not be extremely smooth. (Thus c_0^{**} is not extremely smooth, for example.) The use of theorem 3.1.6 arises because it is usually easier to establish an inequality than an equality. For example, we have lemma 2.6.4, as well as

LEMMA 3.1.7: If X^{**} is strongly extremely rotund, then $||I - P_1|| \leq 1$.

Proof. If $||I - P_1|| > 1$, then from Brown's theorem there exists $x^{**} \in S^{**}$ with

$$||x^{**} - Q_0^{**}(x^{**})|| > 1.$$

Let $\varepsilon > 0$ and choose $y^* \in S^*$ such that

$$x^{**}(y^*) > 1 - \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} 2 &\geq ||x^{**} + Q_0^{**}(x^{**})|| \\ &\geq (x^{**} + Q_0^{**}(x^{**}))(y^*) \\ &= 2x^{**}(y^*) \\ &> 2 - \varepsilon, \end{aligned}$$

from which we conclude that $||x^{**} + Q_0^{**}(x^{**})|| = 2$. But

$$\text{dist}(x^{**} - Q_0^{**}(x^{**}), X^{**}) = ||Q_0^{**}(x^{**})|| = 1,$$

so that $||x^{**} - Q_0^{**}(x^{**})|| \neq \text{dist}(x^{**} - Q_0^{**}(x^{**}), X^{**})$. Hence X^{**} is not strongly extremely rotund. ▨

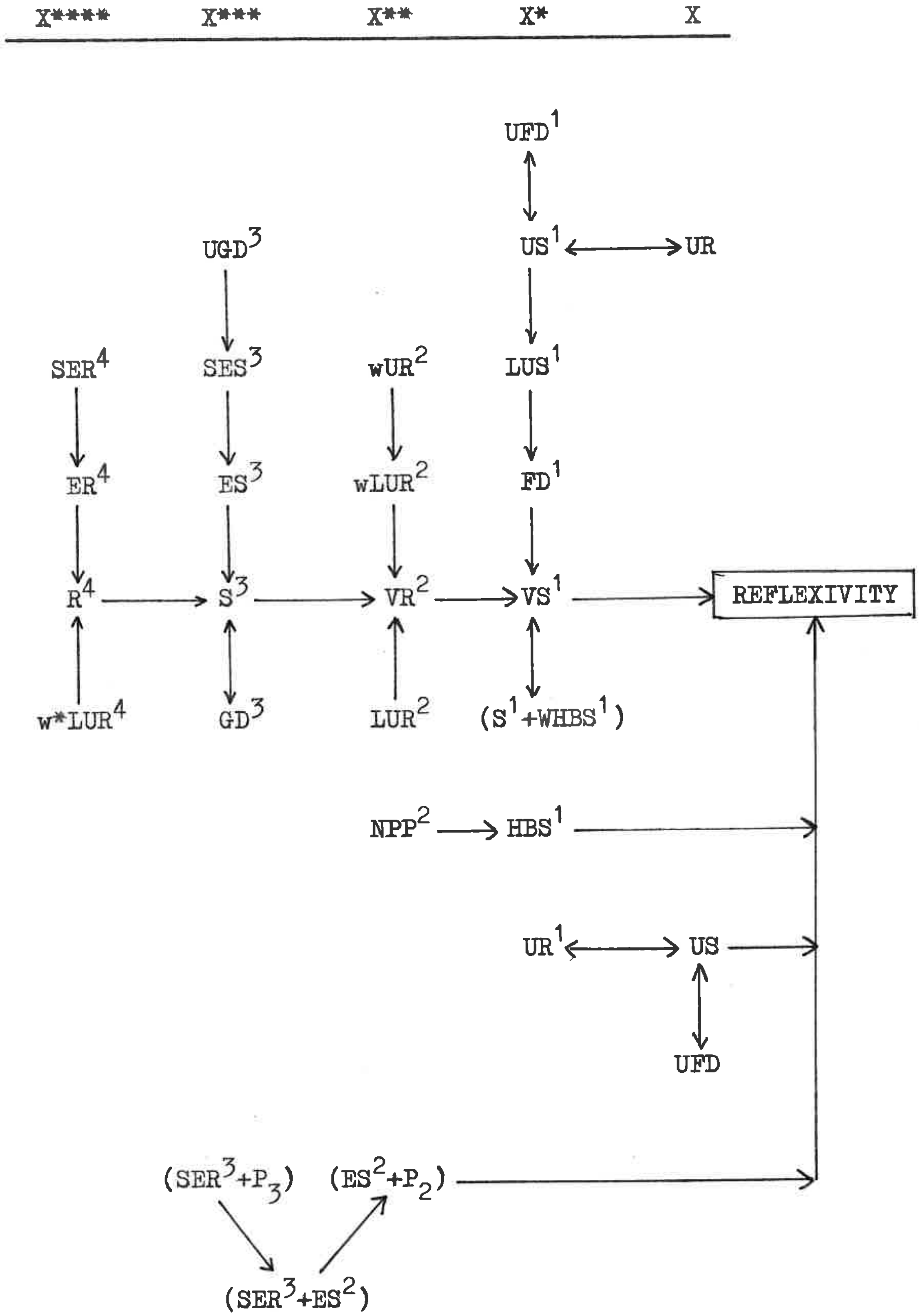
Using this lemma and proposition 2.6.5, we obtain the following corollaries of theorem 3.1.6:

THEOREM 3.1.8: If X^{***} is strongly extremely rotund and X** is extremely smooth, then X is reflexive.

THEOREM 3.1.9: If X*** is strongly extremely rotund and $||I - P_3|| \leq 1$, then X is reflexive (where $P_3 := Q_3Q_2^*$.)

3.2 Chart of implications for reflexivity

The results of chapters 2 and 3 are interrelated as illustrated by the following chart of implications for reflexivity. We use the same abbreviations as before, and we write simply P_n for the condition $||I - P_n|| \leq 1$. The geometrical properties are grouped in columns according to the conjugate space in which they apply.


X****
X***
X**
X*
X
UFD1
UGD3
US1
UR
SER4
SES3
wUR2
LUS1
ER4
ES3
wLUR2
FD1
R4
S3
VR2
VS1
REFLEXIVITY
w*LUR4
GD3
LUR2
(S1+WHBS1)
NPP2
HBS1
UR1
US
UFD
(SER3+P3)
(ES2+P2)
(SER3+ES2)

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