Arc Length Contest

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To enliven our discussion of arc length problems, I often challenge the students in my calculus course to an arc length contest. Each student (or team of students) is asked to find three examples of a continuous function f that satisfies

1. $f(x) \geq 0$ on the interval $0 \leq x \leq 1$;

2. $f(0) = 0, \quad f(1) = 0$;

3. the area bounded by the graph of f and the x-axis between $x = 0$ and $x = 1$ is equal to 1.

The student must compute the arc length for each of her three functions, and the winner of the contest is the one who has the function with the smallest arc length on the unit interval. This is an interesting open-ended problem for most students since they do not have much experience in constructing functions that satisfy required conditions. I am often surprised by some of the ingenuity students use in coming up with their contest entries. The contest requires evaluation of two integrals for each function, one for the area and one for the arc length. Moreover, it is important for the students to recognize when one of their functions gives rise to an improper integral for the arc length since such an integral may be difficult to approximate accurately. Thus the contest provides an opportunity to combine several topics from the typical calculus syllabus.

The shortest closed curve enclosing an area of 1 is a circle of radius $1/\sqrt{\pi}$. Its arc length is $2\sqrt{\pi}$. Since a unit area is bounded by any curve in our contest together with the interval $[0, 1]$, the length of a contest curve cannot be less than $2\sqrt{\pi} - 1 = 2.5449$. If one imagines an ideal inextensible string of length $L > 1$ nailed down at the points $(0, 0)$ and $(1, 0)$, and pushed outward by a uniform pressure, then the equilibrium curve (where the force due to the pressure is balanced by the tension in the string in each small arc) has constant curvature in any portion where it is smooth, so it is a circular arc. This suggests that an arc of a circle joining the points $(0, 0)$ and $(1, 0)$ will yield the smallest arc length for a curve satisfying the three conditions, as illustrated in figure 1. To achieve the area of 1, we would need

$$1 = \pi r^2 - 2 \cdot \frac{1}{2} r^2 \theta + 2 \cdot \frac{1}{4} s = \left(s^2 + \frac{1}{4}\right)\left(\pi - \arctan\left(\frac{1}{2s}\right)\right) + \frac{1}{2} s$$

which yields a center for the circle at $(0.5, 0.35487517)$ with a radius of $r = 0.6131365$ and an arc length of 2.6831. This arc, whose length is within approximately 0.14 of the lower bound $2\sqrt{\pi} - 1$, satisfies the three requirements, but unfortunately is not the graph of a function and thus would violate the rules of the contest.

Imagine now a situation where we have vertical walls at $x = 0$ and $x = 1$, a floor at $y = 0$, and a "rubber band" stretched between $(0, 0)$ and $(1, 0)$. Inject a unit volume of fluid under the
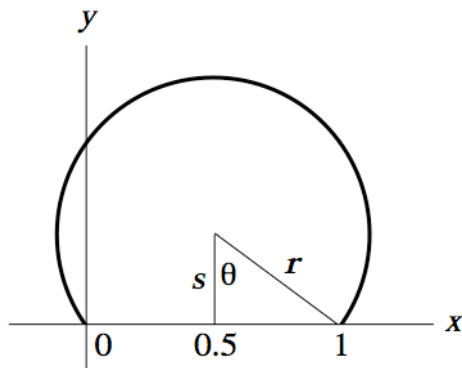
Figure 1: Arc of circle enclosing area of 1

rubber band. The region will fill to a semicircle and then extend upwards, keeping a semicircular cap. Without the vertical walls we would just get the circular bubble of Figure 1. The semicircle of radius $1/2$ has area $\pi/8$. An enclosed area of 1 could thus be obtained by extending the semicircle upwards by a height of $1 - \pi/8$, in essence placing the semicircle atop a rectangle with base along the unit interval. The two side walls of the rectangle and the arc of the semicircle would then have a total length of $2 + \pi/4 = 2.7853981634$. Unfortunately, this is not the graph of a function, either! However, we can get arbitrarily close to this length with a continuous, piecewise-defined function by leaning the vertical walls in by an amount ε and using a semicircle of radius $1/2 - \varepsilon$. This suggests that $2 + \pi/4$ is the greatest lower bound of the lengths of admissible curves.

The piecewise function just described has yet to be entered in the contest by any of my students, however. The entries have always consisted of combinations of the typical functions encountered in calculus: polynomials, radicals, trigonometric functions, and logarithms. There are two obser-vations that greatly aid in constructing functions for the contest. First, students quickly realize after an appropriate suggestion from the instructor that any continuous function g that satisfies conditions (1) and (2) can be modified to obtain a function f satisfying all three conditions by dividing g by the value of its definite integral from $x = 0$ to $x = 1$.

Less obvious, however, is the fact that the best functions to consider are those that are symmetric about the line $x = 1/2$. Indeed, if the function f satisfies conditions (1)–(3) and we take $g(x) = \frac{1}{2}(f(x) + f(1 - x))$, then g will also satisfy conditions (1)–(3), will be symmetric about the line $x = 1/2$, and will have a smaller arc length on $[0, 1]$ than f. To see why this holds, observe that the graph of $H(x) = \sqrt{1 + x^2}$ is concave up over the real line. Therefore the line segment between any two points on the graph lies above the graph, which means that $H\left(\frac{1}{2}a + \frac{1}{2}b\right) \leq \frac{1}{2}H(a) + \frac{1}{2}H(b)$ for any a and b. If f is differentiable on [0,1], apply this inequality with $a = f'(x)$ and $b = -f'(1 - x)$ for each x in [0,1] and integrate over the interval to conclude that

$$\int_0^1 \sqrt{1 + g'(x)^2}\,dx \;\leq\; \frac{1}{2}\int_0^1 \sqrt{1 + f'(x)^2}\,dx + \frac{1}{2}\int_0^1 \sqrt{1 + f'(1 - x)^2}\,dx$$
$$= \int_0^1 \sqrt{1 + f'(x)^2}\,dx$$

If f is only differentiable on the open interval (0,1), but the improper arc length integral for f converges, then the inequality is still valid by considering limits. The typical functions entered in the contest are differentiable on at least the open unit interval.

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The most familiar family of functions for many students are polynomials. The only quadratic polynomial eligible for the contest is $f(x) = 6x(1 - x)$, which has an arc length of 3.24903. Many students will also investigate cubic polynomials, but by our second observation above, this is a losing proposition since if f is any cubic polynomial satisfying the contest conditions, then $g(x) = \frac{1}{2}(f(x) + f(1-x)) = 6x(1-x)$. Therefore no eligible cubic polynomial has an arc length less than 3.24903.

There is hope for a fourth degree polynomial, however. Let $f(x) = Ax + Bx^2 + Cx^3 + Dx^4$. Conditions (2) and (3) yield two linear equations for the four coefficients $A, B, C,$ and D that can be reduced to

$$A = \frac{1}{10}(60 + 5C + 8D)$$
$$B = -\frac{3}{10}(20 + 5C + 6D)$$

By our observation we will assume that f is symmetric about the line $x = 1/2$. Then

$$0 = f(x) - f(1-x) = (C + 2D)x\left(1 - 3x + 2x^2\right)$$

for all x in the unit interval. Therefore $C = -2D$. We know that f has roots at $x = 0$ and $x = 1$. The other potential roots would be at

$$x = \frac{5D \pm \sqrt{5D(120 + D)}}{10D}$$

To satisfy condition (1) of the contest, these two roots must be complex or, if real, must either lie outside the open unit interval or be a double root at $x = \frac{1}{2}$ (which occurs when $D = -120$). Investigating these possibilities reveals that condition (1) holds when $-120 \leq D \leq 30$. Now the arc length of the polynomial f will be a function of the coefficient D. Figure 2 shows a graph of this arc length function for $-120 \leq D \leq 30$. We see there is a minimum that Mathematica determines to be 2.95752 at $D = -22.4935$. Therefore the best 4th degree polynomial for the contest is $f(x) = 10.4987x - 32.9922x^2 + 44.987x^3 - 22.4935x^4$ with an arc length of 2.95752.

One could continue to investigate higher degree polynomials to find functions with arc length even less than 2.95752. For example, the 6th degree polynomial $16.2511x - 99.5369x^2 + 316.046x^3 - 531.71x^4 + 448.424x^5 - 149.475x^6$ satisfies the conditions of the contest and has an arc length of 2.8796. Such investigations become increasingly more difficult, however, because the arc lengths become functions of more than one coefficient after invoking requirements (2) and (3) and the symmetry assumption to find linear relationships among the coefficients.

Besides polynomials, the other most popular functions the students use in the contest are those that start with the sine function and a semicircle. After adjusting for condition (3), these two choices lead to the entries $f(x) = \frac{\pi}{2}\sin(\pi x)$ with an arc length of 3.3655, and $f(x) = \frac{8}{\pi}\sqrt{x - x^2}$, the upper half of an ellipse with an arc length of 2.91946. The semi-ellipse has always won the contest, but just barely. This year one group of students decided to investigate functions of the form $f(x) = A_n x^n \arccos(x)$ for $n > 0$. Their three entries consisted of the functions with $n = 1/100$, $n = 1/2$, and $n = 1$. Based on their numerical approximations, they conjectured that the arc length could be made smaller by taking n closer to 0. Unfortunately, they failed to realize the numerical difficulties in evaluating the improper integral for $n = 1/100$ (with $A_n = 1.013095$). Derive, the computer algebra system we use in class, gives a "dubious accuracy" warning and a value of 1.97361 for the arc length using the default precision! However, the graph of this function is concave down
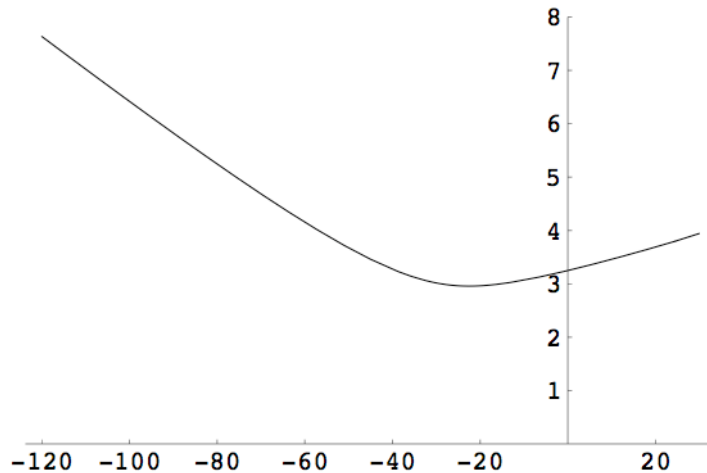
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Figure 2: Arc length of 4th degree polynomial as function of coefficient D

as suggested by Figure 3, a behavior that can be confirmed by computing the second derivative. The graph has a maximum at approximately (0.015, 1.51). The actual arc length will therefore be greater than the sum of the lengths of the line segment connecting the origin with this maximum point and the segment connecting the maximum point with $(1, 0)$. This sum is 3.313. The arc length for $n = 1/2$, with $A_n = 1.716209468$, is 2.91913 which just barely beats the semi-ellipse.

Could this group of students have done better? Using Mathematica to integrate $x^n \arccos(x)$ on the interval $[0, 1]$ reveals that we should take $A_n = 2^{n+2} \Gamma \left((3 + n)/2 \right)^2 / \left(\pi \Gamma(n + 1) \right)$ to have the function $f(x) = A_n x^n \arccos(x)$ satisfy condition (3) of the contest. After substituting this expression, the arc length of f over the unit interval becomes a function of n. Figure 4 shows that this arc length function has a minimum that Mathematica computes to occur at $n = 0.379358$, yielding an arc length of 2.90467.

The function we just found is not symmetric, however. If we let

$$g(x) = 0.76618 \left(x^{0.379358} \arccos(x) + (1 - x)^{0.379358} \arccos(1 - x) \right)$$

we obtain a function that is symmetric and has arc length 2.87535. This is within 3.23% of the greatest lower bound of the lengths of admissible curves. The best solution that I have found, though, that is built from the standard functions of calculus is $f(x) = 1.10278 \left[\sin(\pi x) \right]^{0.153764}$ with an arc length of 2.78946. The reason this works so well is that the small exponent makes the sides of the graph very steep near $x = 0$ and $x = 1$ so that the graph approximates quite closely the semicircle sitting atop the rectangle of height $1 - \pi/8$ as shown in Figure 5. Anyone have a better entry for the contest?

Students interested in the arc length contest might also be interested in reading about the history of similar types of problems. Dido's problem involves finding the figure with the greatest area among all the plane figures bounded by a line and a curve of given length. The arc length contest, posed in an equivalent minimization form, requires that the curve be the graph of a function and that the figure be bounded by the unit interval on the x-axis. Finding the closed plane curve of given length enclosing the maximum area is the classic isoperimetric problem. The isoperimetric problem, however, places no restrictions on the shape of the figures involved. These, and more general types of problems, can be studied with the calculus of variations.
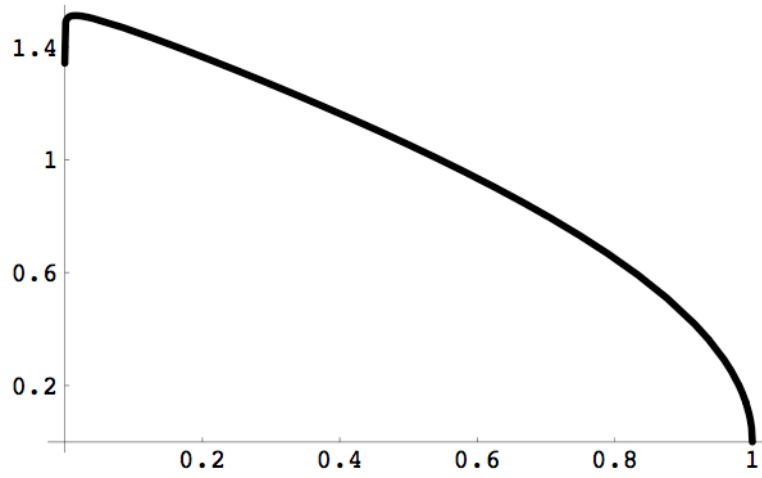
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Figure 3: Graph of $f(x) = A_n x^n \arccos(x)$ for $n = 1/100$, $A_n = 1.013095$
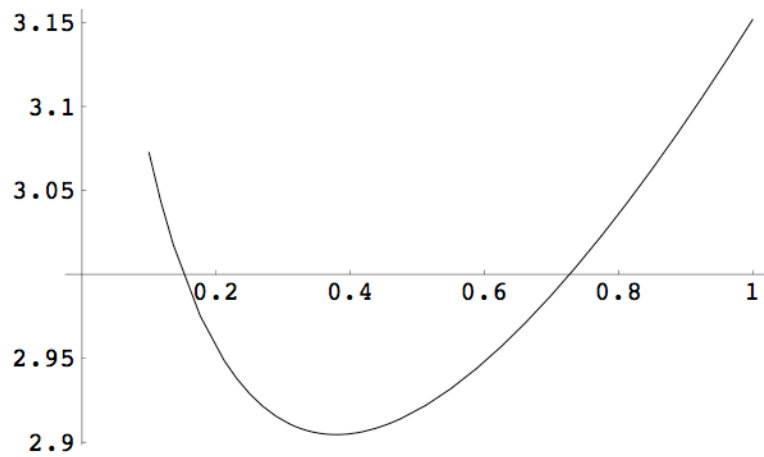

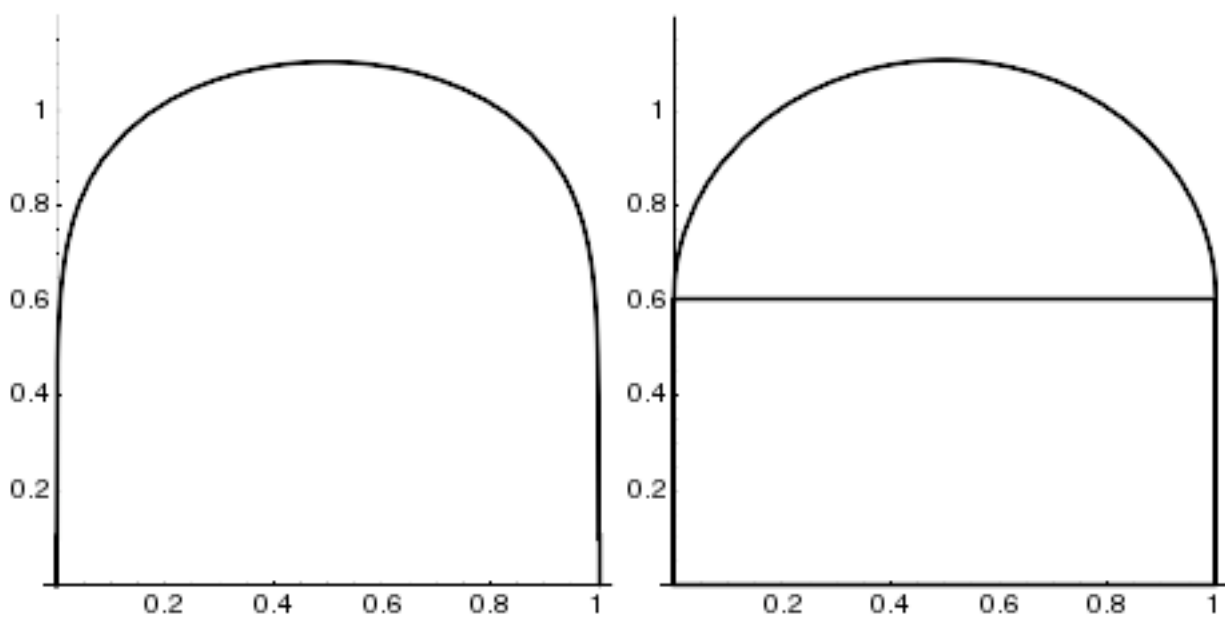Figure 4: Arc length of $A_n x^n \arccos(x)$ as a function of n

Figure 5: Graph of $1.10278[\sin(\pi x)]^{0.153764}$ and the minimal semicircle.

The top 10 student entries in the arc length contest (as of 1997):

	Function	Arc Length
(10)	$-4x^3 + 4x$	3.31707
(9)	$-\frac{12}{5}\left(x^3 + x^2 - 2x\right)$	3.27402
(8)	$9.1440276\left(2^x - x^2 - 1\right)$	3.25382
(7)	$-6x^2 + 6x$	3.24903
(6)	$10x\left(1 - \sqrt{x}\right)$	3.22108
(5)	$-4x\ln(x)$	3.2136
(4)	$\frac{15}{4}x\sqrt{1-x}$	3.17617
(3)	$\frac{8}{\pi}x\arccos(x)$	3.1518
(2)	$\frac{8}{\pi}\sqrt{x - x^2}$	2.91946
(1)	$1.716209468\sqrt{x}\arccos(x)$	2.91913