Approximating the Sum of a Convergent Series

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The BC Calculus Course Description mentions how technology can be used to explore convergence and divergence of series, and lists various tests for convergence and divergence as topics to be covered. But no specific mention is made of actually estimating the sum of a series, and the only discussion of error bounds is for alternating series and the Lagrange error bound for Taylor polynomials. With just a little additional effort, however, students can easily approximate the sum of many common convergent series and determine how precise that approximation will be.

Approximating the Sum of a Positive Series

Here are two methods for estimating the sum of a positive series whose convergence has been established by the integral test or the ratio test. Some fairly weak additional requirements are made on the terms of the series. Proofs are given in the appendix.

Let $S = \sum_{n=1}^{\infty} a_n$ and let the nth partial sum be $S_n = \sum_{k=1}^{n} a_k$.

1. Suppose $a_n = f(n)$ where the graph of f is positive, decreasing, and concave up, and the improper integral $\int_1^\infty f(x)\,dx$ converges. Then

$$S_n + \int_{n+1}^{\infty} f(x)\,dx + \frac{a_{n+1}}{2} < S < S_n + \int_{n}^{\infty} f(x)\,dx - \frac{a_{n+1}}{2}. \tag{1}$$

   (If the conditions for f only hold for $x \ge N$, then inequality (1) would be valid for $n \ge N$.)

2. Suppose (a_n) is a positive decreasing sequence and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$.

   - If $\frac{a_{n+1}}{a_n}$ decreases to the limit L, then

$$S_n + a_n\left(\frac{L}{1-L}\right) < S < S_n + \frac{a_{n+1}}{1 - \frac{a_{n+1}}{a_n}} \tag{2}$$

   - If $\frac{a_{n+1}}{a_n}$ increases to the limit L, then

$$S_n + \frac{a_{n+1}}{1 - \frac{a_{n+1}}{a_n}} < S < S_n + a_n\left(\frac{L}{1-L}\right). \tag{3}$$

Example 1: $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$

The function $f(x) = \dfrac{1}{x^2}$ is positive with a graph that is decreasing and concave up for $x \geq 1$, and $a_n = f(n)$ for all n. In addition, $\int_1^\infty f(x)\,dx$ converges. This series converges by the integral test. By inequality (1),

$$S_n + \frac{1}{n+1} + \frac{1}{2(n+1)^2} < S < S_n + \frac{1}{n} - \frac{1}{2(n+1)^2}. \tag{4}$$

This inequality implies that S is contained in an interval of width

$$\frac{1}{n} - \frac{2}{2(n+1)^2} - \frac{1}{n+1} = \frac{1}{n(n+1)^2}.$$

If we wanted to estimate S with error less than 0.0001, we could use a value of n with $\frac{1}{n(n+1)^2} < 0.0002$ and then take the average of the two endpoints in inequality (4) as an approximation for S. The table feature on a graphing calculator shows that $n = 17$ is the first value of n that works. Inequality (4) then implies that $1.6449055 < S < 1.6450871$[1] and a reasonable approximation would be

$$S \approx \frac{1.6449055 + 1.6450870}{2} \approx 1.645$$

to three decimal places. With $n = 100$, inequality (4) actually shows that $1.6449339 < S < 1.6449349$, and hence we know for sure that $S = 1.64493\ldots$.

Of course, in this case we actually know that $S = \frac{\pi^2}{6} = 1.644934066\ldots$. Notice also that $S_{100} \approx 1.6349839$, so the partial sum with 100 terms is a poor approximation by itself.

Example 2: $S = \sum_{n=1}^{\infty} \frac{n}{n^4+1}$

Let $f(x) = \frac{x}{x^4+1}$. The graph of f is decreasing and concave up for $x \geq 2$. Also

$$\int_n^\infty \frac{x}{x^4+1} = \frac{\pi}{4} - \frac{1}{2}\arctan(n^2)$$

and so the improper integral converges. We can therefore use inequality (1) for $n \geq 2$, and so

$$S_n + \frac{\pi}{4} - \frac{1}{2}\arctan((n+1)^2) + \frac{n+1}{2((n+1)^4+1)} < S < S_n + \frac{\pi}{4} - \frac{1}{2}\arctan(n^2) - \frac{n+1}{2((n+1)^4+1)}.$$

for $n \geq 2$. Using $n = 10$ in this inequality yields $0.6941559 < S < 0.6942724$. We can conclude that $S \approx 0.694$ to three decimal places.

Example 3: $S = \sum_{n=0}^{\infty} \frac{1}{n!}$

The terms of this series are decreasing. In addition,

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1}$$

[1] We will use the convention for positive endpoints of truncating the left endpoint of the interval and rounding up the right endpoint. This will make the interval slightly larger than that given by the actual symbolic inequality.

which decreases to the limit $L = 0$. By inequality (2)

$$S_n < S < S_n + \frac{\frac{1}{(n+1)!}}{1 - \frac{1}{n+1}} = S_n + \frac{1}{n!n}.$$

for all n. Using $n = 10$ in this inequality yields $2.7182818 < S < 2.7182819$ and hence $S \approx 2.7182818$. These, of course, are the first seven decimal places of $e = 2.718281828\ldots$.

Example 4: $S = \sum_{n=1}^{\infty} \frac{1}{n^2 5^n}$

We have

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{1} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{1}{5}$$

which increases to the limit $L = \frac{1}{5}$. According to inequality (3)

$$S_n + \frac{\frac{1}{(n+1)^2 5^{n+1}}}{1 - \frac{1}{5}\left(\frac{n}{n+1}\right)^2} < S < S_n + \frac{1}{n^2 5^n} \cdot \frac{\frac{1}{5}}{\frac{4}{5}}$$

which simplifies to

$$S_n + \frac{1}{(4n^2 + 10n + 5)\, 5^n} < S < S_n + \frac{1}{4n^2 5^n}.$$

With $n = 5$, this inequality shows that $0.2110037 < S < 0.2110049$.

Example 5: $S = \sum_{n=1}^{\infty} \frac{n!}{n^n}$

We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

which is less than 1 for all n and which decreases to the limit $L = \frac{1}{e}$. From inequality (2) we get (after some simplification)

$$S_n + \frac{n!}{n^n} \frac{1}{e-1} < S < S_n + \frac{n!}{(n+1)^n - n^n}.$$

Using $n = 10$ gives $1.8798382 < S < 1.8798548$.

Approximating the Sum of an Alternating Series

Let $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and let the nth partial sum be $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$. We assume that (a_n) is a positive decreasing sequence that converges to 0.

1. The standard error bound is given by

$$S_n - a_{n+1} < S < S_n + a_{n+1} \tag{5}$$

2. Suppose the sequence defined by $b_n = a_n - a_{n+1}$ decreases monotonically to 0. (One way to achieve this is if $a_n = f(n)$ where f is positive with a graph that is decreasing asymptotically to 0 and concave up.) Then

$$\text{if } S_n < S, \text{ then } S_n + \frac{a_{n+1}}{2} < S < S_n + \frac{a_n}{2}; \tag{6}$$

$$\text{if } S < S_n, \text{ then } S_n - \frac{a_n}{2} < S < S_n - \frac{a_{n+1}}{2}. \tag{7}$$

Both of these can be summarized by the inequality

$$\frac{a_{n+1}}{2} < |S - S_n| < \frac{a_n}{2}.$$

Inequality (5) is credited to Leibniz and is the error bound described in the BC Calculus Course Description. Inequalities (6) and (7) are consequences of a proof published in 1962 by Philip Calabrese, then an undergraduate student at the Illinois Institute of Technology (see reference [2]). Calabrese proved that $|S - S_n| < \epsilon$ if $a_n \leq 2\epsilon$, and that furthermore, if $a_n = 2\epsilon$ for some n, then S_n is the first partial sum within ϵ of the sum S. See the appendix for the derivation of inequalities (6) and (7).

Example 6: $S = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{4}{2n-1}$

This is an alternating series that converges by the alternating series test. If $f(x) = \frac{4}{2x-1}$, then the graph of f is positive, decreasing to 0, and concave up for $x \geq 1$. For odd n, inequality (7) implies that

$$S_n - \frac{2}{2n-1} < S < S_n - \frac{2}{2n+1}. \tag{8}$$

If we wanted to estimate the value of S with error less than 0.0001, the typical method using the error bound from inequality (5) would use a value of n for which $a_{n+1} = \frac{4}{2n+1} < 0.0001$. This would require using 20,000 terms. On the basis of inequality (8), however, we can take as an estimate for S the midpoint of that interval, that is, for odd n,

$$S \approx S_n - \frac{1}{2}\left(\frac{2}{2n+1} + \frac{2}{2n-1}\right) = S_n - \frac{4n}{4n^2-1}, \tag{9}$$

with an error less than half the width of the interval. So for an error less than 0.0001, we only need

$$\frac{1}{2}\left(\frac{2}{2n-1} - \frac{2}{2n+1}\right) = \frac{2}{4n^2-1} < 0.0001.$$

The first odd solution is $n = 71$, just a bit less than 20,000! The estimate from (9) using $n = 71$ is $S \approx 3.1415912$, with error less than 0.0001. Since $S = \pi$, this estimate is actually within 1.4×10^{-6} of the true value. By the way, the partial sum S_{71} is approximately 3.1556764.

Example 7: $S = \sum_{n=0}^{\infty} (-1)^n \frac{18^n}{(2n)!}$

This is an alternating series that converges by the alternating series test. Let $b_n = a_n - a_{n+1}$. It is not obvious that the sequence b_n decreases monotonically to 0. An investigation with the table feature of a graphing calculator, however, suggests that this is true for $n \geq 3$. We can therefore use inequality (6) when n is an odd integer greater than 3 (note that inequality (6) holds for odd n's because this series starts with $n = 0$.) Hence

$$S_n + \frac{1}{2}\frac{18^{n+1}}{(2n+2)!} < S < S_n + \frac{1}{2}\frac{18^n}{(2n)!} \qquad \text{for odd } n \geq 3.$$

With $n = 9$ we can estimate that S lies in the interval $(-0.4526626, -0.4526477)$[2], an interval of length 1.49×10^{-5}. But wait, we can actually do better than this! Since the terms of this series decrease so quickly because of the factorial in the denominator, we actually have $a_{n+1} < \frac{1}{2}a_n$ for $n \geq 3$. So if we combine inequalities (5) and (6), we can deduce that for this series,

$$S_n + \frac{1}{2}\frac{18^{n+1}}{(2n+2)!} < S < S_n + \frac{18^{n+1}}{(2n+2)!} \qquad \text{for odd } n \geq 3.$$

Now $n = 9$ gives the interval $(-0.4526626, -0.4526618)$ containing the value of S, an interval of length 8×10^{-7}. (Note: What is the exact sum of this series?)

References

[1] Bart Braden, Calculating Sums of Infinite Series, *The American Mathematical Monthly*, Vol. 99, No. 7. (Aug.–Sep., 1992), 649–655.

[2] Philip Calabrese, A Note on Alternating Series, *The American Mathematical Monthly*, Vol. 69, No. 3. (Mar., 1962), 215–217. (Reprinted in *Selected Papers on Calculus*, The Mathematical Association of America, 1968, 353–354.)

[3] Rick Kreminski, Using Simpson's Rule to Approximate Sums of Infinite Series, *The College Mathematics Journal*, Vol. 28, No. 5. (Nov., 1997), 368–376.

[4] R.K. Morley, The Remainder in Computing by Series, *The American Mathematical Monthly*, Vol. 57, No. 8. (Oct., 1950), 550–551.

[5] R.K. Morley, Further Note on the Remainder in Computing by Series, *The American Mathematical Monthly*, Vol. 58, No. 6. (Jun.–Jul., 1951), 410–412.

Appendix

Proof of Inequality (1)

Let $S = \sum_{n=1}^{\infty} a_n$ and let $S_n = \sum_{k=1}^{n} a_k$. Suppose $a_n = f(n)$ where the graph of f is positive, decreasing to 0, and concave up, and the improper integral $\int_1^{\infty} f(x)\,dx$ converges. The series converges by the integral test. Because the graph is concave up, the area of the shaded trapezoid of width 1 shown in Figure (1) is greater than the area under the curve. Therefore

[2]For negative endpoints, round down the left endpoint and truncate the right endpoint.

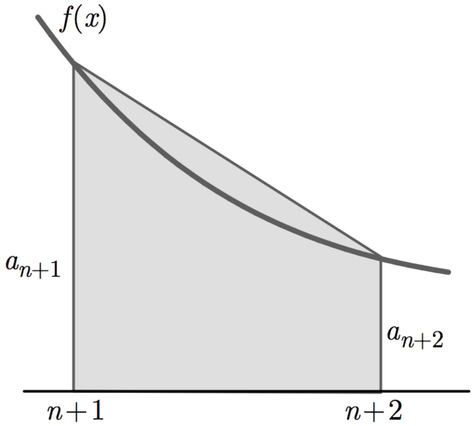

Figure 1

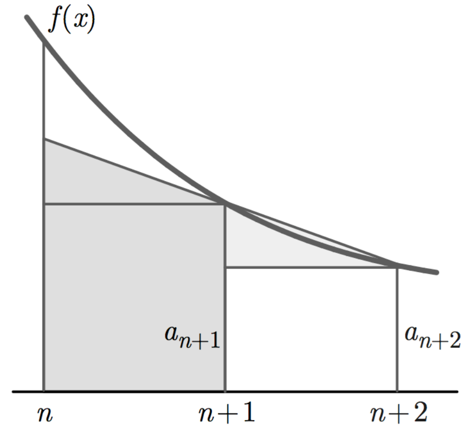

Figure 2

$$\int_{n+1}^{n+2} f(x)\,dx < \frac{1}{2}\left(a_{n+1} + a_{n+2}\right).$$

Hence

$$\begin{aligned}\int_{n+1}^{\infty} f(x)\,dx &< \frac{1}{2}\left(a_{n+1} + a_{n+2}\right) + \frac{1}{2}\left(a_{n+2} + a_{n+3}\right) + \frac{1}{2}\left(a_{n+3} + a_{n+4}\right) + \cdots \\ &= \frac{1}{2}a_{n+1} + a_{n+2} + a_{n+3} + \cdots \\ &= S - S_n - \frac{1}{2}a_{n+1}\end{aligned}$$

In Figure (2), the graph of f lies above that tangent line at $x = n+1$ (because of the positive concavity) and therefore also lies above the continuation of the secant line between $x = n+1$ and $x = n+2$. This implies that the area of the shaded trapezoid in Figure (2) of width 1 between $x = n$ and $x = n+1$ is less than the area under the curve, and so

$$\int_{n}^{n+1} f(x)\,dx > a_{n+1} + \frac{1}{2}\left(a_{n+1} - a_{n+2}\right).$$

Hence

$$\begin{aligned}\int_{n}^{\infty} f(x)\,dx &> a_{n+1} + \frac{1}{2}\left(a_{n+1} - a_{n+2}\right) + a_{n+2} + \frac{1}{2}\left(a_{n+2} - a_{n+3}\right) + a_{n+3} + \frac{1}{2}\left(a_{n+3} - a_{n+4}\right) + \cdots \\ &= \frac{1}{2}a_{n+1} + a_{n+1} + a_{n+2} + a_{n+3} + \cdots \\ &= \frac{1}{2}a_{n+1} + S - S_n\end{aligned}$$

Proof of Inequalities (2) and (3)

Let $S = \sum_{n=1}^{\infty} a_n$ and let $S_n = \sum_{k=1}^{n} a_k$. Suppose (a_n) is a positive decreasing sequence and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, where the ratios decrease to L. The series converges by the ratio test.

Let $r = \dfrac{a_{n+1}}{a_n} < 1$. Then $\dfrac{a_{k+1}}{a_k} < r$ for all $k \geq n$. Hence

$$\begin{aligned}
&a_{n+1} < a_n r \\
&a_{n+2} < a_{n+1} r < a_n r^2 \\
&a_{n+3} < a_{n+2} r < a_n r^3 \\
&\qquad \vdots
\end{aligned}$$

We therefore conclude that

$$S - S_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{n+k} < \sum_{k=1}^{\infty} a_n r^k = \frac{a_n r}{1-r} = \frac{a_{n+1}}{1 - \frac{a_{n+1}}{a_n}}.$$

But we also have $L < \dfrac{a_{k+1}}{a_k}$ for all $k \geq n$. By a similar argument as above,

$$S - S_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{n+k} > \sum_{k=1}^{\infty} a_n L^k = a_n \frac{L}{1-L}.$$

Combining these two results gives inequality (2). A similar argument for the inequalities with r and L reversed proves inequality (3).

Proof of Inequalities (6) and (7)

Let $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ and let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$, where (a_n) is positive decreasing sequence that converges to 0. Let $b_n = a_n - a_{n+1}$, where we assume that the sequence (b_n) also decreases monotonically to 0. Then

$$S = S_n + (-1)^n \left(b_{n+1} + b_{n+3} + b_{n+5} + \cdots\right)$$

and

$$S = S_{n-1} + (-1)^{n+1} \left(b_n + b_{n+2} + b_{n+4} + \cdots\right).$$

Because the sequence (b_n) decreases,

$$|S - S_n| = b_{n+1} + b_{n+3} + b_{n+5} + \cdots < b_n + b_{n+2} + b_{n+4} + \cdots = |S - S_{n-1}|.$$

Therefore $|S - S_n| < |S - S_{n-1}|$. Similarly, $|S - S_{n+1}| < |S - S_n|$. But S lies between the successive partial sums, so it follows that

$$a_n = |S_n - S_{n-1}| = |S - S_n| + |S - S_{n-1}| > 2\,|S - S_n|$$

and

$$a_{n+1} = |S_{n+1} - S_n| = |S - S_{n+1}| + |S - S_n| < 2\,|S - S_n|.$$

Combining these two results shows that

$$\frac{a_{n+1}}{2} < |S - S_n| < \frac{a_n}{2}$$

from which inequalities (6) and (7) can be obtained.