Spectral Theory

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Spectral Theory (1/21)

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I. General Spectral Theory in a B-Space
Riesz-Dunford opr Calc.

II. Compact and Riesz opr

III. Spectral operator (- Dunford)

IV. Decomposable Opr (Colojoară - Foias)

Ch. V - General Spectra Thy in B-Spaces

$\mathfrak{X}$ complex B-Space, $B(\mathfrak{X})$ bounded linear oprs on $\mathfrak{X}$.

5.1.1 Defn - Topologies on $B(\mathfrak{X}, \mathfrak{Y})$

(a) The uniform (or norm) topology: induced by $T \mapsto \|T\|$

(b) The strong opr topology : 0-nhds of the form $N(x_1, \dots x_n) := \{ T : \|Tx_i\| \leq 1 \}$
or $T_\alpha \xrightarrow{SOT} T \iff T_\alpha x \underset{\|\cdot\|}{\longrightarrow} Tx \quad \forall x \in \mathfrak{X}$.

(c) The weak opr topology: 0-nhds of the form $N(x_1, \dots, x_n, y_1^*, \dots, y_m^*) := \{T : |y_i^* T x_j| \leq 1\}$

or $T_\alpha \xrightarrow{WOT} T \iff y^* T_\alpha x \longrightarrow y^* T x \quad \forall x \in \mathfrak{X}, \ \forall y^* \in \mathfrak{Y}^*$

5.1.2 <u>Theorem</u>

If a lin fcnal on $B(\mathfrak{X}, \mathfrak{Y})$ is cont. in the strong opr topology $\iff$ it is cont. in the weak opr top

<u>proof</u>

See D.S.

5.1.3 <u>Defn</u>

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be normed spaces and T a lin. opr in $B(\mathfrak{X}, \mathfrak{Y})$. The <u>adjoint</u> (conjugate, dual) of T is the opr $T^* \in B(\mathfrak{Y}^*, \mathfrak{X}^*)$ defined by

$$(T^* y^*)(x) = y^*(Tx)$$

Notation $x^*(x) =: \langle x, x^* \rangle$, so the above is

$$\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle$$

but <u>WARNING</u>

$(\lambda T)^* = \lambda T^*$ not $\lambda^* T^*$ as in the Hilbert Space setting.

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5.1.4 Theorem

The map $T \mapsto T^*$ is an isometric isomorphism of $B(\mathfrak{X}, \mathfrak{Y})$ into $B(\mathfrak{Y}^*, \mathfrak{X}^*)$.

proof

clearly isomorphism.

$$\|T^*\| = \sup_{\|y^*\| \le 1} \|T^* y^*\| = \sup_{\substack{\|y^*\| \le 1 \\ \|x^*\| \le 1}} \|T^* y^* x\|$$

$$= \sup_{\|x^*\| \le 1} \|y^* T x\| = \sup_{\|x^*\| \le 1} \|Tx\| = \|T\|$$

$\boxtimes$

5.1.5 Theorem

Let $\mathfrak{X}$ and $\mathfrak{Y}$ and $\mathfrak{Z}$ be normed lin. spaces. Then

(a) $T \in B(\mathfrak{X}, \mathfrak{Y})$ $U \in B(\mathfrak{Y}, \mathfrak{Z}) \Rightarrow UT \in B(\mathfrak{X}, \mathfrak{Z})$ and

$(UT)^* = T^* U^*$.

(b) $T^{**} \in B(\mathfrak{X}^{**}, \mathfrak{Y}^{**})$ is an extension of $T \in B(\mathfrak{X}, \mathfrak{Y})$ in the sense that $T^{**} J_{\mathfrak{X}} = J_{\mathfrak{Y}} T$. ($J_{(\cdot)}$ canonical map)

(c) If $\mathfrak{X}$ and $\mathfrak{Y}$ are B-Spaces, then T has a bdd inverse $\Longleftrightarrow$ T^* has a bdd inverse in which case $(T^{-1})^* = (T^*)^{-1}$.

proof

(b) $(T^{**} J_{\mathfrak{X}} x) y^* = J_{\mathfrak{X}} x (T^* y^*) = (T^* y^*)(x) = y^*(Tx)$

$= (J_{\mathfrak{Y}} Tx)(y^*)$

(c) $\Rightarrow$ if $U = T^{-1}$, $UT = I$ $TU = I$ then $T^* U^* = I^* = I$

and $U^* T^* = I^* = I$

$\Leftarrow$ If T^* has a bdd inv then T^{**} does by $\Rightarrow$

$\therefore$ T^{**} is a lin isom of $\mathfrak{X}^{**}$ onto $\mathfrak{Y}^{**}$, is 1-1 and maps $J\mathfrak{X}$ into a closed subspace of $\mathfrak{Y}^{**}$.

Since $T^{**}\mathfrak{X} = J T$, T is 1-1 and has closed range in $\mathfrak{Y}$.

claim $T\mathfrak{X} = \mathfrak{Y}$. if not $T\mathfrak{X}$ is closed and $T\mathfrak{X} \subsetneq \mathfrak{Y}$

and by H.B. $\exists\ y^* \neq 0 \ni y^* T x = 0 \ \forall x \in \mathfrak{X}$

$\Rightarrow (T^* y^*) x = 0$

$\Rightarrow T^* y^* = 0$ contradicting T^* is 1-1. $\boxtimes$

<u>Coro</u>

$\mathfrak{X}$ a B-Space and $T \in B(\mathfrak{X})$ then $\rho(T) = \rho(T^*)$,

$\sigma(T) = \sigma(T^*)$ and $R(\lambda; T^*) = (R(\lambda; T))^*$

5.1.6 <u>Theorem</u>

$\mathfrak{X}$ and $\mathfrak{Y}$ B-Spaces. a lin map $T: \mathfrak{X} \to \mathfrak{Y}$

is norm-norm cont $\Leftrightarrow$ it is ω-ω cont.

<u>proof</u>

$\Rightarrow$ $N_1 = \{ y \in \mathfrak{Y} : |y^* y| \leq 1, \ y^* \in F \}$

For $y^* \in F$, let $x^* = T^* y^*$ and set

$$N_2 = \{ x \in \mathfrak{X} : |x^* x| \le 1, \ x^* = T^* y^*, \ y^* \in F \}$$

Note $T(N_2) = N_1$

$\Leftarrow$ if $y^* \in \mathcal{Y}^*$ then $y^* \circ T$ is a w-cont fcnal

so $y^* T \in \mathfrak{X}^* \Rightarrow y^* T$ is cont. in the norm top.

OMT $\Rightarrow$ T is norm-norm cont.

$\blacksquare$

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5.1.7 THEOREM: Let $\mathfrak{X}, \mathcal{Y}$ be B-spaces.

(a) Let $T \in B(\mathfrak{X}, \mathcal{Y})$. If $S = T^*$, then S is w^*-w^* cont.

(b) If $S : \mathcal{Y}^* \longrightarrow \mathfrak{X}^*$ is linear and w^*-w^* cont., then $\exists\, T \in B(\mathfrak{X}, \mathcal{Y})$ s.t. $S = T^*$

Proof (a) Suppose $y^*_\alpha \to y^*$ w^* in $\mathcal{Y}^*$. Then

$\forall x \in \mathfrak{X}$ $\quad Sy^*_\alpha(x) = T^* y^*_\alpha(x) = y^*_\alpha(Tx) \to y^*(Tx) = Sy^*(x)$

and so $Sy^*_\alpha \to Sy^*$ w^*.

(b) Fix $x \in \mathfrak{X}$. The map $y^* \mapsto (Sy^*)x$ is a w^*-continuous linear functional on $\mathcal{Y}^*$. Hence $\exists\, y_x$ s.t.

$$y^* y_x = (Sy^*)x$$

for all $y^* \in \mathcal{Y}^*$. Define $T : \mathfrak{X} \to \mathcal{Y}$ by $Tx := y_x$. T is linear

Claim: T is w-w continuous.

If $x_\alpha \to x$ weakly, then

$$y^* T x_\alpha = (Sy^*)x_\alpha \to (Sy^*)x = y^*(Tx)$$

Hence $Tx_\alpha \to Tx$ weakly.

Therefore T is norm-norm continuous, i.e. $T \in \mathcal{B}(\mathfrak{X}, \mathcal{Y})$. Clearly $T^* = S$.

$\blacksquare$

§5.2 Annihilators and Ranges

5.2.1. **DEFINITION:** (a) Let $\mathfrak{X}$ be a normed linear space. If $A \subset \mathfrak{X}$

$$A^\perp := \{x^* \in \mathfrak{X}^* : x^* A = 0\}$$

"annihilator of A"

(b) Let $B \subset \mathfrak{X}^*$. Then

$$B_\perp := \{x \in \mathfrak{X} : Bx = 0\}$$

"preannihilator of B"

Note: $B_\perp = \kappa^{-1}(\kappa\mathfrak{X} \cap B^\perp)$

5.2.2. **LEMMA:** Let $A \subset \mathfrak{X}$, $B \subset \mathfrak{X}^*$

(a) $A^\perp$ is a closed subspace of $\mathfrak{X}^*$

(b) $B_\perp$ is a closed subspace of $\mathfrak{X}$

(c) $A \subset (A^\perp)_\perp$; in fact $\overline{\text{sp}}\, A = (A^\perp)_\perp$

(d) $B \subseteq (B_\perp)^\perp$; in fact $\overline{sp}\, B \subseteq (B_\perp)^\perp$

Proof (a) $A^\perp$ is actually w^*-closed. If $x_\alpha^* \longrightarrow x^*$ w^*, and $x_\alpha^* \in A^\perp$, then

$$0 = x_\alpha^*(x) \longrightarrow x^*(x) \quad \forall x \in A$$

$$\Rightarrow x^* \in A^\perp$$

(c) Let $x \in A$, $x^* \in A^\perp$, so $x^*(x) = 0$. Then $x \in (A^\perp)_\perp$
Hence $A \subset (A^\perp)_\perp \Rightarrow \overline{sp}\, A \subset (A^\perp)_\perp$
If $x_0 \notin \overline{sp}(A)$, then by the Hahn-Banach theorem $\exists\, x_0^*$ s.t.

$$x_0^*(x_0) = 1 \qquad x_0^*(\overline{sp}(A)) = 0$$

Then $x_0^* \in A^\perp$, but $x_0^*(x_0) \neq 0$, so $x_0 \notin (A^\perp)_\perp$.
Hence $\overline{sp}(A) = (A^\perp)_\perp$

$\blacksquare$

5.2.3. THEOREM: (a) If $\mathfrak{M}$ is a norm closed subspace of $\mathfrak{X}$, then $\mathfrak{M} = (\mathfrak{M}^\perp)_\perp$

(b) If $\mathfrak{N}$ is a normed closed subspace of $\mathfrak{X}^*$, then $\mathfrak{N} \subset (\mathfrak{N}_\perp)^\perp$ (If $\mathfrak{X}$ is reflexive we get equality)

5.2.4. **THEOREM:** Let $\mathcal{X}, \mathcal{Y}$ be normed linear space, $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$

(a) $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp} = \overline{\mathcal{R}(T)}^{\perp}$

(b) $\mathcal{N}(T) = \mathcal{R}(T^*)_{\perp}$

(c) $\overline{\mathcal{R}(T)} = \mathcal{N}(T^*)_{\perp}$

(d) $\overline{\mathcal{R}(T^*)} \subset \mathcal{N}(T)^{\perp}$

Proof (a) $y^* \in \mathcal{N}(T^*) \Leftrightarrow T^* y^* = 0 \Leftrightarrow T^* y^* x = 0 \ \forall x \in \mathcal{X}$

$\Leftrightarrow y^*(Tx) = 0 \ \forall x \in \mathcal{X}$

$\Leftrightarrow y^* R(T) = 0$

$\Leftrightarrow y^* \in \mathcal{R}(T)^{\perp}$

(b) same

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Example: Let $\mathcal{X} = \mathcal{Y} = \ell_1$ and define

$$T(x_1, x_2, \ldots) = \left(\tfrac{1}{1}x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \ldots\right)$$

Then $\eta(T) = \{0\} \Rightarrow \eta(T)^\perp = \mathcal{X}^* = \ell_\infty$. Note that

$$T^*(y_1, y_2, \ldots) = \left(\tfrac{1}{1}y_1, \tfrac{1}{2}y_2, \ldots\right)$$

Claim: $T^*(\ell_\infty) = \mathcal{R}(T^*)$ is separable in ℓ_∞. But ℓ_∞ is not separable, and so $\overline{\mathcal{R}(T^*)}$ is a proper subspace of ℓ_∞. Hence

$$\overline{\mathcal{R}(T^*)} \subsetneq \eta(T)^\perp$$

5.2.5. COROLLARY $\mathcal{R}(T)$ dense $\Leftrightarrow T^*$ is 1-1
$\mathcal{R}(T^*)$ dense $\Rightarrow T$ is 1-1

Proof: a) Let $y^* \in \eta(T^*)$. Then $T^*y^* = 0$, so

$$T^*y^*(x) = 0 \ \forall x \Rightarrow y^*(Tx) = 0 \ \forall x$$

$$\Rightarrow y^*(\mathcal{R}(T)) = 0 \Rightarrow y^*(\mathcal{Y}) = 0$$

$$\Rightarrow y^* = 0$$

b) $\mathcal{R}(T^*)$ dense $\Rightarrow$ $\mathcal{X}^* = \overline{\mathcal{R}(T^*)} \subset \mathcal{N}(T)^\perp$

$$\Rightarrow \mathcal{N}(T) = (\mathcal{N}(T)^\perp)_\perp = (\mathcal{X}^*)_\perp = \{0\}$$

$\blacksquare$

Recall $\sigma(T^*) = \sigma(T)$

5.2.6. **THEOREM** $T \in \mathcal{B}(\mathcal{X})$

(a) $\sigma_p(T) \subset \sigma_p(T^*) \cup \sigma_r(T^*)$

(b) $\sigma_c(T^*) \subset \sigma_c(T) \subset \sigma_c(T^*) \cup \sigma_r(T^*)$

(c) $\sigma_r(T) \subset \sigma_p(T^*) \subseteq \sigma_p(T) \cup \sigma_r(T)$

(d) $\sigma_r(T^*) \subset \sigma_p(T) \cup \sigma_c(T)$

(e) If $\mathcal{X}$ is reflexive $\sigma_c(T) = \sigma_c(T^*)$, $\sigma_r(T^*) \subset \sigma_p(T)$

Proof (a) $\lambda \in \sigma_p(T) \Rightarrow \mathcal{N}(\lambda - T) \neq 0 \Rightarrow \overline{\mathcal{R}(\lambda - T^*)} \subsetneq \mathcal{X}^*$ and so $\lambda \notin \sigma_c(T^*) \Rightarrow \lambda \in \sigma_p(T^*) \cup \sigma_r(T^*)$.

(b) $\lambda \in \sigma_c(T^*) \Leftrightarrow \begin{cases} \lambda - T^* & \text{1-1} \\ \lambda - T^* & \text{has dense proper range} \end{cases}$

$$\Rightarrow \begin{cases} \mathcal{R}(\lambda - T) & \text{dense} \\ \mathcal{N}(\lambda - T) = 0 \end{cases} \Rightarrow \lambda \in \sigma_c(T)$$

Conversely,

$$\lambda \in \sigma_c(T) \Leftrightarrow \begin{cases} \lambda - T & 1-1 \\ \mathcal{R}(\lambda - T) & \text{dense} \end{cases} \Rightarrow \begin{cases} ? & \\ \lambda - T^* & 1-1 \end{cases}$$

and so $\lambda \notin \sigma_p(T^*)$

The rest of the proofs are similar.

$\blacksquare$

Remark - If $\mathfrak{X}$ is not reflexive, then $\sigma_c(T) \cap \sigma_r(T^*)$ can be non-empty. See last example, where $0 \in \sigma_c(T) \cap \sigma_r(T^*)$

5.2.7. <u>THEOREM</u> (BANACH) Let $\mathfrak{X}, \mathcal{Y}$ be Banach spaces. If $T \in B(\mathfrak{X}, \mathcal{Y})$ and $\mathcal{R}(T)$ is closed, then $\mathcal{R}(T^*)$ is closed. Moreover,

$$\mathcal{R}(T) = \eta(T^*)_\perp \; ; \; \mathcal{R}(T^*) = \eta(T)^\perp$$

i.e.

$$\mathcal{R}(T) = \{ y \in \mathcal{Y} : T^* y^* = 0 \Rightarrow y^* y = 0 \}$$

$$\mathcal{R}(T^*) = \{ x^* \in \mathfrak{X}^* : Tx = 0 \Rightarrow x^* x = 0 \}$$

Proof. See D-S

5.28. **THEOREM** (BANACH) If $\mathcal{R}(T^*)$ is closed, then $\mathcal{R}(T)$ is closed

Proof (hard) See D-S VI 6.4
Kaufman PAMS 17 (1966)
Yosida Functional Analysis (1965 ed.) p 205

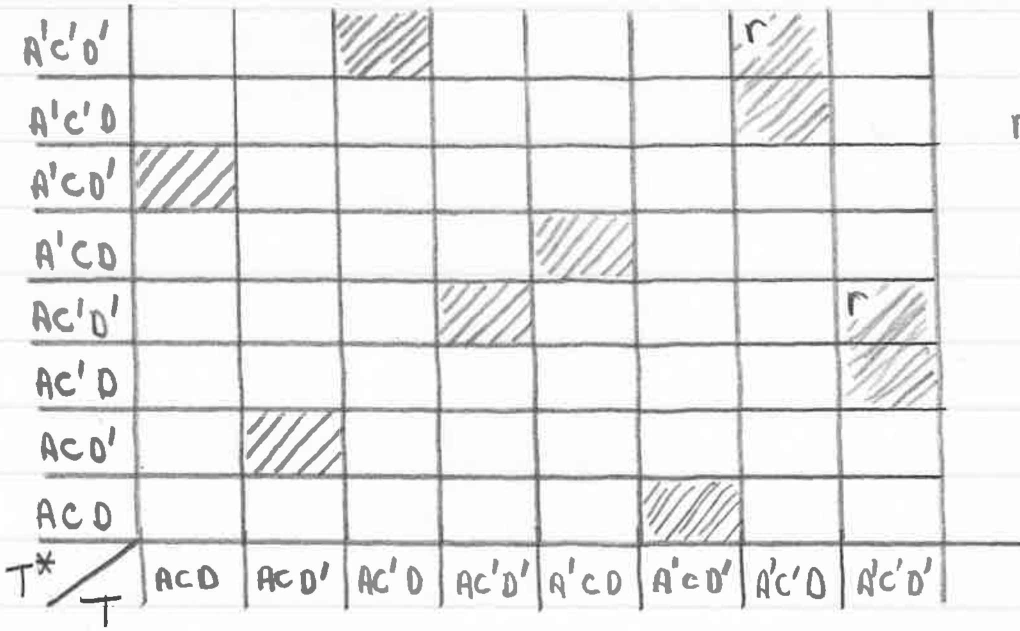
The ACD classification $T \in B(\mathfrak{X}, \mathcal{Y})$

A = annihilates a non-zero vector $\qquad$ A′ = 1-1
C = range is closed $\qquad$ C′ = range not closed
D = range is dense $\qquad$ D′ = range not dense

$$\left[\begin{array}{l} \sigma_c = A'C'D \\ \sigma_r = A'D' \\ \sigma_p = A \\ \rho = A'CD \quad \text{(resolvent)} \end{array}\right.$$

J.T. Joichi (1959)

(See next page)



r = cannot occur if $\mathfrak{X}$ is reflexive (even if $\mathcal{Y} = \ell_2$)

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$\lambda \in \sigma_a(T)$ means $\exists \, \|x_n\| = 1$ s.t. $(\lambda - T)x_n \to 0$

approximate spectrum

$\lambda \in \sigma_a(T) \Leftrightarrow \exists \, m > 0$ s.t. $\|(\lambda - T)x\| \geq m\|x\| \quad \forall x$

$\Rightarrow \mathcal{R}(\lambda - T)$ is closed

$\sigma_p(T) \cup \sigma_c(T) \subseteq \sigma_a(T) \qquad \partial\sigma(T) \subseteq \sigma_a(T)$

$\sigma_a(T)$ is closed but there is no obvious relationship between $\sigma_a(T)$ and $\sigma_a(T^*)$ (e.g. left shift operator)

Let $\mathcal{X}_1, \mathcal{X}_2$ be B-spaces, $T_i \in B(\mathcal{X}_i)$. Let $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$. Define

$$T(x_1, x_2) := (T_1 x_1, T_2 x_2)$$

5.2.9. **THEOREM**: $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$

Proof $\lambda \in \rho(T_1) \cap \rho(T_2)$. Define

$$A(x_1, x_2) := (R(\lambda, T_1)x_1, R(\lambda, T_2)x_2)$$

Then $A(\lambda - T) = I = (\lambda - T)A$, and so $\lambda \in \rho(T)$. Hence

$$\sigma(T) \subset \sigma(T_1) \cup \sigma(T_2)$$

Let $\lambda \in \rho(T)$. We want to show that $\lambda \in \rho(T_1)$

$$(\lambda - T)(x_1, x_2) = ((\lambda - T_1)x_1, (\lambda - T_2)x_2)$$

Let $J_1 : \mathfrak{X}_1 \to \mathfrak{X}$ and $\pi_1 : \mathfrak{X} \to \mathfrak{X}_1$ be given by

$$J_1(x_1) := (x_1, 0)$$
$$\pi_1(x_1, x_2) := x_1$$

Then $\pi_1 J_1 = I_1$. Observe that

$$(\pi_1 R(\lambda, T) J_1)(\lambda - T_1)x_1 = \pi_1 R(\lambda, T)((\lambda - T_1)x_1, 0)$$

$$= \pi_1 R(\lambda, T)(\lambda - T)(x_1, 0)$$

$$= \pi_1 (x_1, 0) = x_1$$

Since

$$\pi_1 (\lambda - T)(x_1, x_2) = (\lambda - T_1) x_1$$

$$= (\lambda - T_1) \pi_1 (x_1, x_2)$$

we have

$$(\lambda - T_1)\pi_1 R(\lambda;T)J_1 = \pi_1(\lambda - T)R(\lambda;T)J_1$$

$$= \pi_1 J_1 = I_1$$

Therefore $R(\lambda;T_1) = \pi_1 R(\lambda;T)J_1$, $\lambda \in \rho(T_1)$.
Similarly, $\lambda \in \rho(T_1) \cap \rho(T_2)$, so $\rho(T) \subset \rho(T_1) \cap \rho(T_2)$
Therefore

$$\sigma(T_1) \cup \sigma(T_2) \subset \sigma(T)$$

5.2.10 <u>LEMMA</u>: If $T \in B(\mathfrak{X})$ and $A \in B(\mathfrak{X}, \mathcal{Y})$ is invertible, then $S := ATA^{-1} \in B(\mathcal{Y})$ and

$$\sigma(S) = \sigma(T)$$
$$R(\lambda;S) = AR(\lambda;T)A^{-1}$$

5.2.11 THEOREM: Let $\mathcal{Y}$ be a B-space, $T \in B(\mathcal{Y})$
Let $\mathcal{Y}_1, \mathcal{Y}_2$ reduce T

$$[\![\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2,\ T\mathcal{Y}_i \subseteq \mathcal{Y}_i]\!]$$

Then $\sigma(T) = \sigma(T|_{\mathcal{Y}_1}) \cup \sigma(T|_{\mathcal{Y}_2})$

Proof. Let $T_j = T|_{\mathcal{Y}_j}$. If $A : \mathcal{Y} \to \mathcal{Y}_1 \oplus \mathcal{Y}_2$ is given by

$$A(y_1+y_2) = (y_1, y_2)$$

then A is an isomorphism. Moreover,

$$T = A^{-1}(T_1 \oplus T_2) A$$

$\boxtimes$

Let $\mathfrak{X}$ be a B-space, $\mathcal{Y} < \mathfrak{X}$ closed subspace. Define

$$[x]_{\mathcal{Y}} = \bar{x} = x + \mathcal{Y} = \text{coset} = \{x+y : y \in \mathcal{Y}\}$$

$$[x_1 + x_2] := [x_1 + x_2]$$

$$\alpha [x] := [\alpha x]$$

$$\|[x]\| := \inf\{\|x+y\| : y \in \mathcal{Y}\} = \operatorname{dist}(x, \mathcal{Y})$$

Then we get a Banach space $\mathfrak{X}/\mathcal{Y}$. $\varphi : X \mapsto [x]$ canonical quotient map. $\|\varphi\| \leq 1$

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5.2.12 THEOREM: Let $\mathcal{Y}$ be a closed subspace of a Banach space $\mathcal{X}$. Then the map

$$\psi_1 : \mathcal{X}^* / \mathcal{Y}^\perp \longrightarrow \mathcal{Y}^*$$

defined by

$$\psi_1 [x^*](y) := x^*(y) \quad \forall y \in \mathcal{Y}$$

is an isometric isomorphism.

5.2.13 THEOREM: The map $\psi_2 : (\mathcal{X}/\mathcal{Y})^* \longrightarrow \mathcal{Y}^\perp$ defined by

$$\psi_2(f)(x) = (f \circ \varphi)(x)$$

for every $x \in \mathcal{X}$ is an isometric isomorphism.

$$\left[\!\left[\ \psi_2^{-1} \omega^* [x] := \omega^*(x) \ \right]\!\right]$$

§5.3 ASCENT AND DESCENT

Let $\mathfrak{X}$ be a vector space. $T : \mathfrak{X} \to \mathcal{Y}$ linear. Note that

$$0 \subset \mathcal{N}(T) \subset \mathcal{N}(T^2) \subset \mathcal{N}(T^3) \subset \dots \subset \mathfrak{X}$$

$$\mathfrak{X} \supset \mathcal{R}(T) \supset \mathcal{R}(T^2) \supset \mathcal{R}(T^3) \supset \dots \supset 0$$

5.3.1. THEOREM:

(a) $\eta(T^n) \subset \eta(T^{n+1})$ $\forall n$

$\mathcal{R}(T^n) \supset \mathcal{R}(T^{n+1})$ $\forall n$

(b) If $\eta(T^k) = \eta(T^{k+1})$ for some $k \in \mathbb{N}$, then $\eta(T^k) = \eta(T^n)$ for all $n \geq k$.

(c) If $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$ for some k, then $\mathcal{R}(T^k) = \mathcal{R}(T^n)$ for all $n \geq k$.

Proof (a) Trivial

(b) Suppose $\eta(T^k) = \eta(T^{k+1})$. If $x \in \eta(T^{k+2})$, then $Tx \in \eta(T^{k+1}) = \eta(T^k) \Rightarrow T^k(Tx) = 0 \Rightarrow x \in \eta(T^{k+1})$. Hence $\eta(T^{k+2}) = \eta(T^{k+1})$. Now use induction.

(c) Suppose $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$. If $y \in \mathcal{R}(T^{k+1})$. Then

$y = T^{k+1}u = T(T^k u) = Tx$, where $x \in \mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$.
Hence $x = T^{k+1}z$, so $y = T(T^{k+1}z) = T^{k+2}z$. Hence $z \in \mathcal{R}(T^{k+2})$.
Therefore $\mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2})$. Now use induction.

$\blacksquare$

5.3.2 <u>DEFINITION:</u> The ascent of T ($a(T)$) is the smallest k s.t. $\eta(T^k) = \eta(T^{k+1})$, or $+\infty$ if no such k exists.
The descent of T ($d(T)$) is the smallest integer s.t. $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$, or $+\infty$ if no such k exists.

<u>Example:</u> (5.3.3)

a) $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$

$$\eta(T^n) = \{(x_1, \dots, x_n, 0, 0, \dots)\} \Rightarrow a(T) = +\infty$$

$$\mathcal{R}(T^n) = \mathcal{X} \ \forall n \Rightarrow d(T) = 0$$

b) $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$

$$\eta(T^n) = \{0\} \Rightarrow a(T) = 0$$

$$\mathcal{R}(T^n) = \{(0, 0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)\} \Rightarrow d(T) = +\infty$$

5.3.4. LEMMA: If $a(T) < \infty$ and $d(T) = 0$, then $a(T) = 0$

Proof. Suppose $a(T) > 0$ and $d(T) = 0$. Then T is not 1-1 but is onto. Let $x_1 \neq 0$ be such that $Tx_1 = 0$. $\exists\, x_2$ s.t. $Tx_2 = x_1$. Then $T^2 x_2 = Tx_1 = 0$. Since $x_1 \neq 0$, $x_2 \neq 0$. Hence $\eta(T^2) \neq \eta(T^1)$ since $x_2 \in \eta(T^2) \setminus \eta(T^1)$. Continue by induction to obtain seq. (x_n) s.t.

$$x_n \in \eta(T^n) \setminus \eta(T^{n-1})$$

This shows that $a(T) > 0 \Rightarrow a(T) = +\infty$ (if $d(T) = 0$)

$\boxtimes$

5.3.5 THEOREM: If $a(T)$ and $d(T)$ are finite, then $a(T) = d(T) =: p$. In this case

$$\mathcal{X} = \mathcal{R}(T^p) \oplus \eta(T^p)$$

Moreover, $T|_{\mathcal{R}(T^p)}$ is 1-1 and onto.

Proof. See Dowson or Taylor

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FUNCTIONAL CALCULUS

§6.1 Analytic Operational Calculus

Let $\mathfrak{X} \neq (0)$ be a complex B-space, $T \in B(\mathfrak{X})$.

6.1.1. DEFINITION: Let $\mathfrak{F}(T)$ be the collection of all functionals f which are analytic on some nbhd of $\sigma(T)$

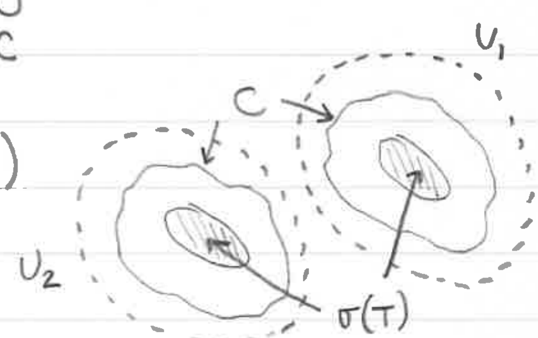
(The nbhds are not necessarily connected and may depend on the functional)

Obviously $\mathfrak{F}(T)$ is a complex vector space.

If C is a "scroc" (simple closed rectifiable oriented contour) in $D_f \cap \rho(T)$ containing $\sigma(T)$ in its interior, define

$$f(T) := \frac{1}{2\pi i} \int_C f(\lambda) R(\lambda; T)\, d\lambda$$

(Bochner integral). Note $f(T) \in B(\mathfrak{X})$



Lemma: If $A \in B(\mathfrak{X}, \mathfrak{Y})$, $f \in \mathcal{F}(T)$, $x^* \in \mathfrak{X}^*$, $x \in \mathfrak{X}$, then

a) $$Af(T) = \frac{1}{2\pi i} \int_C f(\lambda)\, A R(\lambda; T)\, d\lambda$$

b) $$f(T)x = \frac{1}{2\pi i} \int_C f(\lambda) \langle R(\lambda;T), x \rangle\, d\lambda$$

c) $$x^* f(T) x = \frac{1}{2\pi i} \int_C f(\lambda) \langle x^*, R(\lambda;T)x \rangle\, d\lambda$$

Proof: Clear

Corollary: $f(T)$ is independent of the choice of C

Proof: If C_1 is another curve, (homotopic to C) then by the ordinary theory (Cauchy's Theorem)

$$x^* f_1(T) x = \frac{1}{2\pi i} \int_{C_1} f(\lambda) \langle x^*, R(\lambda;T) x \rangle\, d\lambda$$

$$= \frac{1}{2\pi i} \int_C f(\lambda) \langle x^*; R(\lambda;T) x \rangle\, d\lambda = x^* f(T) x$$

for every x^*, x. Hence $f_1(T) = f(T)$.

$\blacksquare$

6.1.2 THEOREM: Let $f, g \in \mathcal{F}(T)$, $\alpha, \beta \in \mathbb{C}$.

(a) $\alpha f + \beta g \in \mathcal{F}(T)$ and $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$

(b) $f \cdot g \in \mathcal{F}(T)$ and $fg(T) = f(T) \circ g(T)$

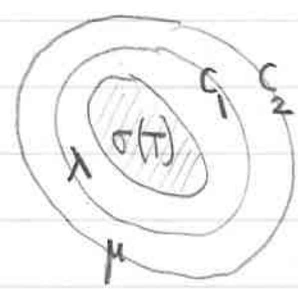
(c) If $f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n$ converges on a nbhd of $\sigma(T)$, then

$$f(T) = \sum_{n=0}^{\infty} c_n T^n \quad (\text{in } B(\mathcal{X}))$$

(d) $f \in \mathcal{F}(T^*)$ and $f(T^*) = f(T)^*$

Proof (b) Note motovation:

$$f(T)g(T) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1} f(\lambda)R(\lambda)\,d\lambda \int_{C_2} g(\mu)\,R(\mu)\,d\mu$$



1st choose C_2
then C_1

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} f(\lambda)\,g(\mu)\,R(\lambda)R(\mu)\,d\lambda\,d\mu$$

[Recall 1st resolvent equation $R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$]

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} f(\lambda) g(\mu) \frac{R(\lambda)}{\mu - \lambda} \, d\lambda \, d\mu$$

$$- \left(\frac{1}{2\pi i}\right)^2 \int_{C_2}\int_{C_1} f(\lambda) g(\mu) \frac{R(\mu)}{\mu - \lambda} \, d\lambda \, d\mu$$

$$= \frac{1}{2\pi i} \int_{C_1} f(\lambda) g(\lambda) R(\lambda) \, d\lambda \; + \; 0$$

since $\displaystyle\int_{C_1} \frac{f(\lambda)}{\mu - \lambda} \, d\lambda = 0$

($\mu - \lambda$ analytic on C_1)

$$= \frac{1}{2\pi i} \int_{C_1} (fg)(\lambda) R(\lambda) \, d\lambda$$

$$= fg(T)$$

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Proof of $fg(T) = f(T)g(T)$

$$g(T)x = \frac{1}{2\pi i}\int_{C_2} g(\mu)\, R(\mu)x\, d\mu$$

$$\Rightarrow \quad x^*R(\lambda)g(T)x = \frac{1}{2\pi i}\int_{C_2} g(\mu)\, x^*R(\lambda)R(\mu)\,x\, d\mu$$

$$= \frac{1}{2\pi i}\int_{C_2} g(\mu)\,\frac{x^*R(\lambda)x}{\lambda-\mu}\,d\mu - \frac{1}{2\pi i}\int_{C_2} g(\mu)\,\frac{x^*R(\mu)x}{\lambda-\mu}\,d\mu$$

$$= g(\lambda)\,x^*R(\lambda)x - \frac{1}{2\pi i}\int_{C_2} g(\mu)\,\frac{x^*R(\mu)x}{\lambda-\mu}\,d\mu$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_1} f(\lambda)\, x^* R(\lambda) g(T) x \, d\lambda = \frac{1}{2\pi i}\int_{C_1} f(\lambda) g(\lambda)\, x^* R(\lambda) x \, d\lambda$$

$$- \left(\frac{1}{2\pi i}\right)^2 \int_{C_1} f(\lambda) \left[\int_{C_2} g(\mu) \frac{x^* R(\mu) x}{\lambda - \mu} d\mu \right] d\lambda$$

$$= x^* fg(T) x - \frac{1}{(2\pi i)^2} \int_{C_2} f(\lambda) \left[\underbrace{\int_{C_1} g(\mu) \frac{x^* R(\mu) x}{\lambda - \mu} d\lambda}_{=0} \right] d\mu$$

$\frac{1}{\lambda - \mu}$ is analytic inside and on C_1

$$= x^* fg(T) x$$

But

$$x^* f(T)[g(T) x] = \frac{1}{2\pi i} \int_{C_1} f(\lambda)\, x^* R(\lambda) [g(T) x] \, d\lambda$$

$$= x^* fg(T) x$$

Hence $f(T) g(T) = fg(T)$.

(c) Suppose $f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n$ converges on a nbhd of $\sigma(T)$.
Then we want to show

$$f(T) = \sum_{n=0}^{\infty} c_n T^n$$

First note that

$$R(\lambda) = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \qquad |\lambda| > r(T)$$

Let $C = \{|\lambda| = r(T)+\varepsilon\}$.

$$\lambda^n R(\lambda) = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1-n}} \quad \text{(converges unif. on compact sets)}$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_C \lambda^n R(\lambda)\,d\lambda = \frac{1}{2\pi i}\int_C \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1-n}}\,d\lambda$$

$$= \sum_{k=0}^{\infty} T^k \frac{1}{2\pi i}\int_C \frac{1}{\lambda^{k+1-n}}\,d\lambda$$

$$= T^n$$

If ε is sufficiently small, $f(\lambda) = \sum c_n \lambda^n$ converges uniformly on C. Hence

$$f(T) = \frac{1}{2\pi i}\int_C f(\lambda)R(\lambda)\,d\lambda = \sum c_n \frac{1}{2\pi i}\int_C \lambda^n R(\lambda)\,d\lambda$$

$$= \sum c_n T^n \qquad \square$$

6.1.3. COROLLARY: If $f, g \in \mathcal{F}(T)$, then

$$f(T)\,g(T) = g(T)\,f(T)$$

and $f(T)$ commutes with T, $R(\lambda, T)$ and with any operator $A \in B(\mathfrak{X})$ which commutes with T.

Proof. If $A(\lambda - T) = (\lambda - T)A$, then $R(\lambda; T)A = A\,R(\lambda; T)$ for all $\lambda \in \rho(T)$

$\blacksquare$

Note if $\lambda \in \rho(T)$

$$R(\lambda; T) = \frac{1}{2\pi i}\int_C \frac{R(\xi; T)}{\lambda - \xi}\,d\xi$$

for

$$\lambda - T = \frac{1}{2\pi i}\int_C (\lambda - \xi)\,R(\xi; T)\,d\xi$$

and so

$$(\lambda - T)\left[\frac{1}{2\pi i}\int_C \frac{R(\xi; T)}{\lambda - \xi}\,d\xi\right] = fg(T) = 1(T) = I$$

$$\left(f(\xi) = \lambda - \xi \,,\quad g(\xi) = \frac{1}{\lambda - \xi} \right)$$

6.1.4 CONVERGENCE THEORY Let $(f_n) \subset$ ... uniformly on some nbhd V of $\sigma(T)$ to a function f. Then and

$$f(T) = \lim f_n(T) \quad (\text{in } B(\mathcal{X}))$$

Proof. Clear that $f \in \mathcal{F}(T)$, so $f(T)$ exists

$$\|f(T) - f_n(T)\| = \left\| \frac{1}{2\pi i} \int_C f(\lambda) - f_n(\lambda) R(\lambda)\, d\lambda \right\|$$

$$\leq \frac{1}{2\pi} \sup_{\lambda \in C} |f(\lambda) - f_n(\lambda)| \cdot \max_{\lambda \in C} \|R(\lambda)\| \cdot \ell($$

$$\leq M \sup_{\lambda \in C} |f(\lambda) - f_n(\lambda)| \to 0$$

6.1.5 SPECTRAL MAPPING THEOREM

If $f \in \mathcal{F}(T)$, then $\sigma(f(T)) = f(\sigma(T))$

Proof. Let $\lambda \in \sigma(T)$ Claim: $f(\lambda) \in \sigma(f(T))$. Define

$$g(\xi) = \begin{cases} \dfrac{f(\xi) - f(\lambda)}{\xi - \lambda} & \xi \neq \lambda \\ f'(\lambda) & \xi = \lambda \end{cases} \qquad \xi \in D_f$$

Then $g \in \mathcal{F}(T)$. Note that

$$(\xi - \lambda)\, g(\xi) = f(\xi) - f(\lambda)$$

$\forall\, \xi, \lambda$. Hence

$$\underbrace{(T-\lambda)}_{\substack{\text{This is}\\ \text{not invertible}}} g(T) = \underbrace{f(T) - f(\lambda)}_{\text{Hence this is not invertible}}$$

$$\Downarrow$$

$$f(\lambda) \in \sigma(f(T))$$

Hence $f(\sigma(T)) \subset \sigma(f(T))$

Conversely, if $\mu \notin f(\sigma(T))$, then

$$h(\xi) := \frac{1}{\mu - f(\xi)}$$

(for $\xi \in$ nbhd of $\sigma(T)$) belongs to $\mathcal{F}(T)$.

$$\mu - f(\xi)\; h(\xi) = 1$$

$$\Rightarrow (\mu - f(T))\, h(T) = I$$

$$\Rightarrow \quad h(T) = R(\mu; f(T))$$

$$\Rightarrow \mu \in \rho(f(T)) \Rightarrow \mu \notin \sigma(f(T))$$

$\boxtimes$

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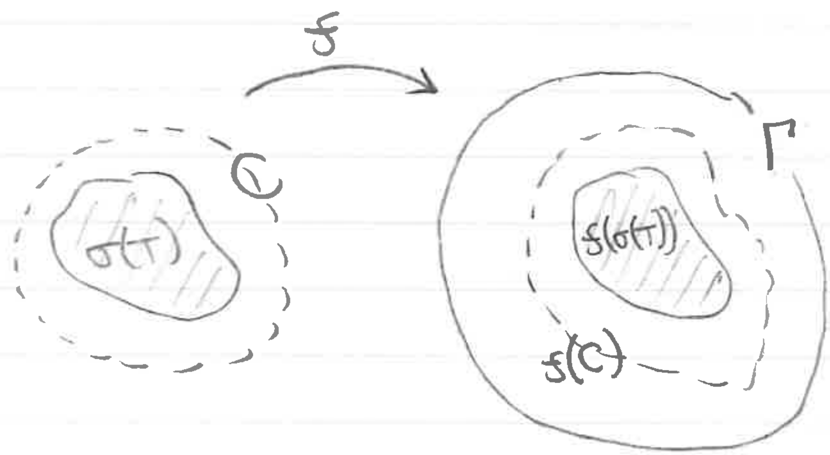
6.1.6. **THEOREM:** Let $f \in \mathfrak{F}(T)$ and $g \in \mathfrak{F}(f(T))$. Then $F := g \circ f \in \mathfrak{F}(T)$ and

$$F(T) = g(f(T))$$

Proof Let Γ be a scroc surrounding $f(\sigma(T)) = \sigma(f(T))$. Then $\exists$ scroc C surrounding $\sigma(T)$ s.t.

$$f(C \cup \operatorname{int} C) \subseteq \operatorname{int} \Gamma$$



If $\lambda \in \Gamma$, then $\lambda \in \rho(f(T))$ and $\lambda - f(\xi) \neq 0$ for $\xi \in C$.
Define

$$A(\lambda) := \frac{1}{2\pi i} \int_C \frac{R(\xi;T)}{\lambda - f(\xi)}\, d\xi$$

Now

$$\lambda - f(T) = \frac{1}{2\pi i} \int_C (\lambda - f(\xi)) R(\xi;T)\, d\xi$$

and so

$$(\lambda I - f(T))\, A(\lambda) = \frac{1}{2\pi i} \int_C R(\xi; T)\, d\xi = I$$

$$\Rightarrow A(\lambda) = R(\lambda; f(T))$$

for $\lambda \in \Gamma$. Therefore

$$g(f(T)) = \frac{1}{2\pi i} \int_\Gamma g(\lambda) R(\lambda; f(T))\, d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma g(\lambda) \left\{ \frac{1}{2\pi i} \int_C \frac{R(\xi;T)}{\lambda - f(\xi)}\, d\xi \right\} d\lambda$$

$$x^* g(f(T))\, x = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma g(\lambda) \left(\int_C \frac{x^* R(\xi;T)\, x}{\lambda - f(\xi)}\, d\xi \right) d\lambda$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_C \left(\int_\Gamma \frac{g(\lambda)}{\lambda - f(\xi)}\, d\lambda \right) x^* R(\xi;T)\, x\, d\xi$$

$$= \frac{1}{2\pi i} \int_C \underbrace{g(f(\xi))}_{F(\xi)}\, x^* R(\xi;T)\, x\, d\xi$$

$$= x^* F(T)\, x$$

$\blacksquare$

Components of the spectrum

Let σ be a clopen subset of $\sigma(T)$. Let

$$f_\sigma := \begin{cases} 1 & \text{on nbhd of } \sigma \\ 0 & \text{on nbhd of } \sigma(T)\setminus\sigma \end{cases}$$

Then $f_\sigma \in \mathcal{F}(T)$. Note that $f_\sigma^2 = f_\sigma$. Let

$$E(\sigma) := f_\sigma(T) = \frac{1}{2\pi i}\int_C R(\lambda;T)\,d\lambda \qquad \sigma \subset \operatorname{int} C,\ \sigma(T)\setminus\sigma \subset \operatorname{ext} C$$

Then $E^2(\sigma) = E(\sigma)$ (If $\sigma = \sigma(T)$, then $E(\sigma) = I$)

6.1.7 THEOREM: Let $\mathfrak{X} \neq (0)$, $T \in B(\mathfrak{X})$. Let Σ_0 be the Boolean algebra of clopen subsets of $\sigma(T)$. Then the map $\sigma \to E(\sigma)$ is an isomorphism of Σ_0 into a Boolean algebra of projections in $B(\mathfrak{X})$, i.e.

(1) $E(\emptyset) = 0$

(2) $E(\sigma(T)) = I$

(3) $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$

(4) $E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2)$

(5) $E(\sigma(T)\setminus\sigma) = I - E(\sigma)$

Moreover, $TE(\sigma) = E(\sigma)T$ and $\sigma(T|E(\sigma)\mathfrak{X}) = \sigma$.

Proof (1) - (6) easy consequence of definition of f_σ. Let $\mathfrak{X}_1 := E(\sigma)\mathfrak{X}$, $\mathfrak{X}_2 = (I - E(\sigma))\mathfrak{X}$. Then

$$\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$$

and T is reduced by $(\mathfrak{X}_1, \mathfrak{X}_2)$. By Theorem 5.2.11

$$(*) \qquad \sigma(T) = \sigma(T|\mathfrak{X}_1) \cup \sigma(T|\mathfrak{X}_2)$$

If $\lambda \notin \sigma$, let

$$h_\lambda(\xi) := \frac{1}{\lambda - \xi}$$

for ξ in a nbhd of σ that does not contain λ

$$h_\lambda(\xi) := 0$$

for ξ in a nbhd of $\sigma(T) - \sigma =: \sigma'$. Then $h_\lambda \in \mathfrak{F}(T)$ and

$$h_\lambda(\xi)(\lambda - \xi) = f_\sigma$$

$$\Rightarrow \quad h_\lambda(T)(\lambda I - T) = E(\sigma)$$

Hence $(\lambda I - T)|\mathfrak{X}_1$ is invertible and has inverse $h_\lambda(T)|\mathfrak{X}_1$. Therefore $\lambda \notin \sigma(T|\mathfrak{X}_1)$, and so $\sigma(T|\mathfrak{X}_1) \subset \sigma$. Also $\sigma(T|\mathfrak{X}_2) \subset \sigma'$. Hence from $(*)$

$$\sigma(T) = \sigma(T|\mathcal{X}_1) \cup \sigma(T|\mathcal{X}_2)$$

$$\subset \sigma_1 \cup \sigma_2 = \sigma(T)$$

$$\Rightarrow \quad \sigma(T|\mathcal{X}_1) = \sigma_1$$
$$\sigma(T|\mathcal{X}_2) = \sigma_2$$

Suppose $E(\sigma) = 0$. Then $E(\sigma)\mathcal{X} = \{0\}$, so

$$\sigma = \sigma(T|E(\sigma)\mathcal{X}) = \sigma(T|\{0\}) = \phi$$

Hence this map $\sigma \mapsto E(\sigma)$ is 1-1.

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Isolated Singularities of $R(\lambda;T)$

Let $\lambda = 0$ be the only point in $\sigma(T)$. $\sigma(T) = \{0\}$ iff T is quasinilpotent ($\|T^n\|^{1/n} \to 0$). Here

$$R(\lambda;T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

for $\lambda \neq 0$. Hence the residue of $R(\lambda;T)$ is I.

6.1.8. THEOREM If $\sigma(T) = \{0\}$, then T is nilpotent of index (= height) $\leq n$ if and only if $\exists M, \delta > 0$ s.t.

$$(*) \qquad \|\lambda^n R(\lambda;T)\| \leq M \quad (0 < |\lambda| < \delta)$$

Proof. Suppose $T^{n-1} \neq 0$ but $T^n = 0$. Then

$$R(\lambda;T) = \frac{I}{\lambda} + \frac{T}{\lambda^2} + \ldots + \frac{T^{n-1}}{\lambda^n}$$

$$\Rightarrow \lambda^n R(\lambda;T) = \lambda^{n-1} I + \ldots + T^{n-1}$$

$$\Rightarrow \|\lambda^n R(\lambda;T)\| \leq M$$

Now suppose $(*)$ holds. Then

$$T^n = \frac{1}{2\pi i} \int_{|\lambda|=\delta} \lambda^n R(\lambda;T)\, d\lambda$$

$$\Rightarrow \|T^n\| \le \frac{1}{2\pi} M\, 2\pi\delta = M\delta$$

(for all δ sufficiently small). Hence $T^n = 0$. ▨

6.1.9 **THEOREM:** Let $\lambda = 0$ be an isolated point of $\sigma(T)$. Let

$$E = \frac{1}{2\pi i} \int_{|\lambda|=\delta} R(\lambda;T)\, d\lambda$$

a) We can write $R(\lambda;T) = R^+(\lambda) + R^-(\lambda) \quad (\lambda \in \rho(T))$ where

$$R^+ R^- = 0 = R^- R^+$$

$R^-(\lambda) := E\,R(\lambda;T)$ and has an analytic extension to $|\lambda| \neq 0$
$R^+(\lambda) := (I-E)\,R(\lambda;T)$ and has an analytic extension to $|\lambda| < \delta$

b) Moreover, $\lambda = 0$ is a pole of order n of $R(\lambda;T)$ iff

$$\|\lambda^n R(\lambda;T)\| \le M \qquad (0 < |\lambda| < \delta)$$

$\|\lambda^{n-1} R(\lambda;T)\|$ not bdd on $0 < |\lambda| < \delta$

i.e. iff $(ET)^{n-1} \neq 0$ but $(ET)^n = 0$.

(c) If $T^k x_0 = 0$, then

$$R(\lambda;T)x_0 = \left(\frac{I}{\lambda} + \frac{T}{\lambda^2} + \dots + \frac{T^{k-1}}{\lambda^k} \right) x_0$$

so that $\|\lambda^k R(\lambda;T)x_0\|$ is bounded on a deleted nbhd of 0.
Moreover $Ex_0 = x_0$.

Conversely, if $\|\lambda^k R(\lambda;T)x_0\|$ is bdd on a deleted nbhd of 0, then $T^k x_0 = 0$.

Proof. E is a projection, commutes with T and $R(\lambda;T)$.
Also

$$\sigma(T|E\mathfrak{X}) = \{0\}$$

Let $T_1 := T|E\mathfrak{X}$, so T_1 is quasi-nilpotent in $B(\mathfrak{X}_1)$, $\mathfrak{X}_1 := E\mathfrak{X}$.
Let $T_2 := T|(I-E)\mathfrak{X}$. Then $\sigma(T_2) = \sigma(T) - \{0\}$.

$$R(\lambda;T) = E\,R(\lambda;T) + (I-E)R(\lambda;T)$$

$$= R^-(\lambda) + R^+(\lambda)$$

Claim: $T_1^n E = T^n E$

Clearly $T_1 E = TE = ETE$ (since E commutes with T)
Hence

$$T_1^{\,2}E = T_1(T_1E) = T_1(ETE) = (T_1E)(TE)$$

$$= TETE = T^2E^2 = T^2E$$

Now use induction.

For $|\lambda| > \|T\|$

$$R(\lambda;T) = \sum \frac{T^n}{\lambda^{n+1}}$$

$$R^-(\lambda) = \sum \frac{T^nE}{\lambda^{n+1}}$$

Since T is quasi-nilpotent,

$$R(\lambda;T_1) = \sum \frac{T_1^{\,n}}{\lambda^{n+1}} \quad (\text{all } \lambda \neq 0)$$

$$R(\lambda;T_1)E = \sum \frac{T_1^{\,n}E}{\lambda^{n+1}} = \sum \frac{T^nE}{\lambda^{n+1}}$$

so $R(\lambda;T_1)E = R^-(\lambda)$ $\forall\, |\lambda| > \|T\|$.

$$R^+(\lambda) = (I-E)R(\lambda;T) = R(\lambda;T_2)(I-E)$$

$$\rho(T_2) \supset \{|\lambda| < \delta\}$$

Hence $R^+(\lambda)$ has analytic extension to $|\lambda| < \delta$

b) If $\lambda = 0$ is a pole of order n of $R(\lambda;T)$, then $\lambda = 0$ is a pole of order n of $R(\lambda;T_1)$ [since $R^+(\lambda)$ is bdd at origin]

$$\Rightarrow \|\lambda^n R(\lambda;T)\| \text{ bdd and } \|\lambda^{n-1} R(\lambda;T)\| \text{ not bdd}$$

c) $|\lambda| > \|T\|$. $\quad R(\lambda;T)x_0 = \sum \dfrac{T^n x_0}{\lambda^{n+1}}$

$$Ex_0 = \frac{1}{2\pi i} \int R(\lambda;T)x_0 \, d\lambda = x_0$$

(since terminates) $\qquad$ residue

As before

$$T^k x_0 = \frac{1}{2\pi i} \int \lambda^k R(\lambda;T)x_0 \, d\lambda$$

Write $x_0 = Ex_0 + (I-E)x_0$

removable singularity

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6.1.10 THEOREM: Let $\lambda = 0$ be a pole of $R(\lambda; T)$ of order p. Then $\lambda = 0$ is an eigenvalue of T and the ascent and descent of T is p. Moreover,

$$\mathcal{N}(T^p) = E\mathfrak{X} \qquad \mathcal{R}(T^p) = E(\sigma(T) - \{0\})\mathfrak{X}$$

Proof. D.S. VII 3.24
Dowson 1.52

6.1.11 THEOREM: If $0 \in \sigma(T)$ and $a(T) = d(T) = p < \infty$, and if $T^p \mathfrak{X}$ is closed, then 0 is a pole of order p.

Proof Dowson 1.54

6.1.12 THEOREM: (Minimal Equation Theorem) Let $f \in \mathcal{F}(T)$, then $f(T) = 0 \Leftrightarrow \forall \lambda \in \sigma(T)$ either

(i) f vanishes in a nbhd of f

or (ii) λ is a pole of $R(\lambda; T)$ of order ν_λ and f has a root of order $\geq \nu_\lambda$ at λ
(Can exist only a finite number of such poles)

Proof VII 3.16 (D.S.)
Dowson 1.41

DEFINITION: $R(\lambda;T)$ is rational if $\exists$ polynomial $Q_1 : \mathbb{C} \to B(\mathfrak{X})$ and polynomial $P_1 : \mathbb{C} \to \mathbb{C}$ s.t.

$$R(\lambda;T) = \frac{Q_1(\lambda)}{P_1(\lambda)}$$

whenever $P_1(\lambda) \neq 0$

6.1.13 **THEOREM:** Let $T \in B(\mathfrak{X})$

(a) $R(\lambda;T)$ is rational $\Leftrightarrow$ there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ of degree ≥ 1 s.t. $p(T) = 0$

(b) There exists a unique monic polynomial of minimal degree s.t. $p(T)=0$ ("minimal polynomial")

(c) If the minimal polynomial has the factorization

$$p(\lambda) = \prod_{j=1}^{k} (\lambda - \lambda_j)^{n_j}$$

then we can write

$$\mathfrak{X} = \sum_{j=1}^{k} \oplus \, \eta\left((T-\lambda_j)^{n_j}\right)$$

(d) If E_j is the projection of $\mathfrak{X}$ onto $\eta(T-\lambda_j)^{n_j}$ and if

$$N := \sum_{j=1}^{k} (T-\lambda_j) E_j$$

then $N^p = 0$ for $p = \max(n_j)$ and

$$T = \sum_{j=1}^{k} \lambda_j E_j + N$$

---

§6.2 Perturbation of Spectra

6.2.1. Lemma: Let $T, S \in B(\mathcal{X})$, and $\lambda \in \rho(T)$. If

$$\| T - S \| \leq \| R(\lambda; T) \|^{-1}$$

then $\lambda \in \rho(S)$ and

$$R(\lambda; S) = R(\lambda; T) \sum_{n=0}^{\infty} \left[(S-T) R(\lambda; T) \right]^n$$

Proof. This follows from thm 3.4.4 (b) since $\| (S-T) R(\lambda; T) \| < 1$

6.2.2. Theorem: If $T \in B(\mathcal{X})$, $\varepsilon > 0$, then there exists $\delta > 0$ such that if $\| S - T \| < \delta$, then

$$\sigma(S) \subset N(\sigma(T), \varepsilon) = \bigcup_{\lambda \in \sigma(T)} D(\lambda, \varepsilon)$$

and

$$\|R(\lambda;S) - R(\lambda;T)\| < \varepsilon \quad \forall \lambda \notin N(\sigma(T), \varepsilon)$$

Proof $\|R(\lambda;T)\| \to 0$ as $|\lambda| \to \infty$. Hence

$$\|R(\lambda;T)\| \leq M_\varepsilon \quad \text{for } \lambda \notin N(\sigma(T), \varepsilon)$$

Let $\delta_1 = 1/M_\varepsilon$. By the lemma, if $\|S-T\| < \delta_1$, then $\sigma(S) \subset N(\sigma(T), \varepsilon)$ and

$$\|R(\lambda;S) - R(\lambda;T)\| \leq \left\| \sum_{n=1}^{\infty} (S-T)^n R(\lambda;T)^n \right\|$$

$$\leq \frac{M_\varepsilon^2 \|S-T\|}{1 - \|S-T\| M_\varepsilon}$$

If $\delta_2 := \dfrac{\varepsilon}{M_\varepsilon^2 + \varepsilon M_\varepsilon}$ and if $\|S-T\| < \delta_2$, then

$$\|R(\lambda;S) - R(\lambda;T)\| < \varepsilon$$

Now choose $\delta = \min(\delta_1, \delta_2)$

$\blacksquare$

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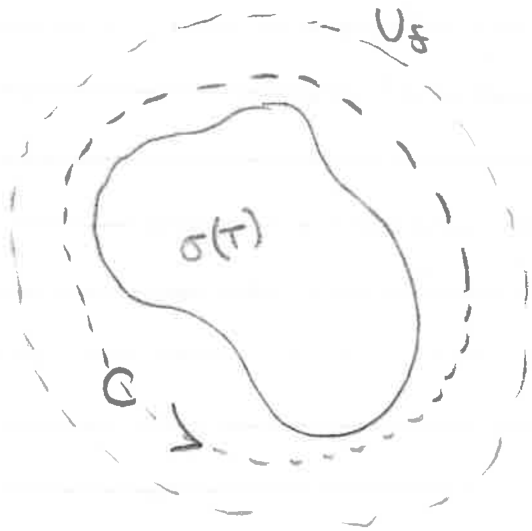
6.2.3. <u>THEOREM:</u> Let $f \in \mathcal{F}(T)$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ s.t. if $\|S - T\| < \delta$, then $f \in \mathcal{F}(S)$ and

$$\| f(S) - f(T) \| < \varepsilon$$

Proof



C scsc $\subset U_f$
$\sigma(T) \subset$ inside C

$\exists\ \delta_1 > 0$ s.t. if $\|S - T\| < \delta_1$, then $\sigma(S) \subset$ inside C (by last theorem) also

$$\| R(\lambda; T) - R(\lambda; S) \| < \varepsilon$$

for all $\lambda \in C$. Hence

$$\| f(S) - f(T) \| = \left\| \frac{1}{2\pi} \int_C \big(R(\lambda; S) - R(\lambda; T)\big) f(\lambda)\, d\lambda \right\|$$

$$\leq \frac{1}{2\pi} \varepsilon \sup_C |f| \cdot \text{length } C$$

$\blacksquare$

6.2.4. DEFINITION: Let $K \subset \mathbb{C}$ be compact and $\varepsilon > 0$.

$$N_\varepsilon(K) = \{\lambda \in \mathbb{C} : |\lambda - \mu| < \varepsilon \text{ for some } \mu \in K\}$$

$$= \bigcup_{\mu \in K} B_\varepsilon(\mu)$$

The Hausdorff distance between two compact sets is

$$H\text{-}d(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset N_\varepsilon(K_2),\ K_2 \subset N_\varepsilon(K_1)\}$$

(F. Hausdorff (Mengenlehre, 3RD Ed., 1935) shows that H-d is a metric on the compact sets of C)

6.2.5. DEFINITION: If (K_n) is a seq. of compact sets in $\mathbb{C}$, define

$$\liminf K_n = \{\lambda \in \mathbb{C} : \text{every nbhd of } \lambda \text{ intersects all but finitely many } K_n\}$$

$$\limsup K_n = \{\lambda \in \mathbb{C} : \text{every nbhd of } \lambda \text{ intersects infinitely many } K_n\}$$

6.2.6. LEMMA: $\liminf K_n$ and $\limsup K_n$ are compact and

$$\liminf_n K_n \subset \limsup_n K_n$$

If $\liminf K_n = \limsup K_n$, then we write $\operatorname{Lim} K_n$ for this common value.

6.2.7. THEOREM (Hausdorff) Let (K_n) be a sequence of compact sets in $\mathbb{C}$

(a) If Hausdorff limit exists, then $\operatorname{Lim} K_n$ exists and equals it.

(b) If $\bigcup_{n=1}^{\infty} K_n$ is bounded in $\mathbb{C}$, and if $\operatorname{Lim} K_n$ exist, then H-limit exists and equals $\operatorname{Lim} K_n$.

6.2.8. DEFINITION: Let τ map a metric space Y into a compact subset of $\mathbb{C}$.

a) We say that τ is upper semicontinuous at y_0 if $y_n \to y_0$ implies $\limsup \tau(y_n) \subset \tau(y_0)$

b) We say that τ is lower semicontinuous at y_0 if $y_n \to y_0$ implies $\tau(y_0) \subset \liminf \tau(y_n)$

c) We say that τ is continuous at y_0 if $y_n \to y_0$ implies $\tau(y_0) = \lim \tau(y_n)$

6.2.9. **THEOREM** The map $T \mapsto \sigma(T)$ is upper semi-continuous at every point $T \in B(\mathfrak{X})$

Proof. Let $T_n \longrightarrow T$ in $B(\mathfrak{X})$. Let $\lambda_0 \notin \sigma(T)$, so $\lambda_0 I - T$ is invertible. Since the invertible operators in $B(\mathfrak{X})$ form an open set, $\exists\, \delta > 0$ s.t. if

$$\| S - (\lambda_0 I - T) \| < \delta$$

then S is invertible. If $|\lambda - \lambda_0| < \tfrac{1}{2}\delta$, then

$$\| (\lambda I - T) - (\lambda_0 I - T) \| < \tfrac{1}{2}\gamma$$

and so $\lambda \in \rho(T)$. If $n \geq n_0$, then $\|T_n - T\| < \tfrac{1}{2}\delta$. Hence

$$\begin{aligned} &\| (\lambda I - T_n) - (\lambda_0 I - T) \| \\ &\leq \| (\lambda I - T_n) - (\lambda I - T) \| + \| (\lambda I - T) - (\lambda_0 I - T) \| \\ &\leq \tfrac{1}{2}\gamma + \tfrac{1}{2}\gamma = \gamma \end{aligned}$$

Hence $\lambda \notin \sigma(T_n)$ $\forall n > n_0$, so $\lambda_0 \notin \limsup \sigma(T_n)$.

$\blacksquare$

6.2.10. Examples

a) $\mathfrak{X} = \ell_2(\mathbb{Z})$. Let (e_m) be usual basis. Define

$$T(\ldots, x_{-1}, (x_0), x_1, \ldots) = (\ldots, x_{-1}, 0, (x_1), x_2, \ldots)$$

$\uparrow$ 0^{th} coord. $\qquad$ $\uparrow$ 0^{th} coord

$$T_n(\ldots, x_{-1}, (x_0), x_1, \ldots) = (\ldots, x_{-1}, \tfrac{1}{n}x_0, (x_1), x_2, \ldots)$$

Then $\|T - T_n\| = 1/n$, so $T_n \to T$. $\|T\| = \|T_n\| = 1$. Hence

$$\sigma(T), \sigma(T_n) \subset \{|\lambda| \leq 1\}$$

Moreover, T_n^{-1} exists and $\|T_n^{-1}\| = n$. In fact $\|(T_n^{-1})^k\| = n$ and so

$$r(T_n^{-1}) = \lim \|(T_n^{-1})^k\|^{1/k} = \lim n^{1/k} = 1$$

Hence $\sigma(T_n^{-1}) \subset \{|\lambda| \leq 1\}$, and so $\sigma(T_n) \subset \{|\lambda| = 1\}$.
We also have $\sigma(T) = \{|\lambda| \leq 1\}$

$$\left((\lambda - T)(\cdots 0, 0, (1), \lambda, \lambda^2, \ldots) = 0 \quad |\lambda| < 1 \right)$$

Hence $\sigma(T) \not\subset \liminf \sigma(T_n)$, so $T \mapsto \sigma(T)$ is not lower semi-continuous.

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Example: (Rickart Banach Algebras, 1960, p 282)

$\mathcal{X} = \ell_2(\mathbb{N})$. For $m \in \mathbb{N}$, let $\alpha_m = e^{-k}$ where k is the highest power of 2 that divides m

$$m = 2^k(2\ell+1)$$

$k = 0, 1, \ldots, \quad \ell = 0, 1, \ldots$

$$T_n e_m := \begin{cases} 0 & \text{if } m = 2^n(2\ell+1) \\ \alpha_m e_{m+1} & \text{otherwise} \end{cases}$$

$$T e_m := \alpha_m e_{m+1}$$

Then $T_n^{2^{n-1}} = 0$, so $\sigma(T_n) = \{0\}$. However $\|T_n - T\| \le e^{-n} \to 0$ so $T_n \to T$, and $r(T) = 1/e^2 > 0$

6.2.11 THEOREM (J.D. Newburgh, Duke Math J. 18 (1951), p165-176)

Let $(T_n) \subset B(\mathcal{X})$, $\|T_n - T\| \to 0$

a) If $\sigma(T)$ is totally disconnected, then $T \mapsto \sigma(T)$ is cont.

(b) If $T_n T = T T_n \ \forall n$, then $T \mapsto \sigma(T)$ is cont.

(c) If (T_n) is a sequence of normal operators in Hilbert space, then $T \mapsto \sigma(T)$ is cont.

CHAPTER VII COMPACT AND RELATED OPERATORS

§7.1 Compact Operators

7.1.1. DEFINITION: $T \in B(\mathcal{X}, \mathcal{Y})$ is compact if $T(S)$ is contained in a compact set in $\mathcal{Y}$

(S = unit ball)

$$\mathcal{K}(\mathcal{X}, \mathcal{Y}) = \text{all compact } T : \mathcal{X} \to \mathcal{Y}$$

7.1.2. LEMMA: T is compact $\Leftrightarrow$ for every bounded sequence (x_n) in $\mathcal{X}$ there is a subsequence (x_{n_k}) such that (Tx_{n_k}) converges in $\mathcal{Y}$ $\Leftrightarrow$ $T(S)$ is totally bounded.

(F. Riesz Acta Math 1918)

7.1.3. Example

(a) $\mathcal{X} = \mathcal{Y} = C[a,b]$ $\quad K : [a,b] \times [a,b] \to \mathbb{C}$ continuous

$$(T_K u)(s) := \int_a^b K(s,t)\, u(t)\, dt \qquad \forall u \in \mathcal{X}$$

Then T_K is compact.

(b) $\mathcal{X} = \mathcal{Y} = L_2[a,b]$ $\quad \iint |K(s,t)|^2\, ds\, dt < \infty$

$$(T_K u)(s) := \int_a^b K(s,t)\, u(t)\, dt \quad \forall u \in \mathcal{X}$$

(Hilbert-Schmidt class)

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7.1.4. **THEOREM:** Let $\mathfrak{X}, \mathcal{Y}$ be B-spaces

(a) If $K \in \mathcal{K}(\mathfrak{X}, \mathcal{Y})$ and $A \in B(\mathcal{Y}, \mathfrak{Z})$, $B \in B(\mathfrak{W}, \mathfrak{X})$, then AK and KB are compact.

(b) If $K_1, K_2 \in \mathcal{K}(\mathfrak{X}, \mathcal{Y})$, then $c_1K_1 + c_2K_2$ is compact.

(c) If $(K_n) \subset \mathcal{K}(\mathfrak{X}, \mathcal{Y})$ and $\|K - K_n\| \to 0$, then K is compact.

(d) If $K \in \mathcal{K}(\mathfrak{X}, \mathcal{Y})$ and $\mathfrak{X}_1$ is a subspace of $\mathfrak{X}$, then $K|_{\mathfrak{X}_1} \in \mathcal{K}(\mathfrak{X}_1, \mathcal{Y})$.

Proof. All immediate (For (c) use Cantor diagonalization)

7.1.5. **COROLLARY:** An operator $F: \mathfrak{X} \to \mathcal{Y}$ with finite dimensional range is compact.

7.1.6. **COROLLARY:** If $\mathfrak{X}$ is a B-space, then $\mathcal{K}(\mathfrak{X})$ is a two sided ideal in $B(\mathfrak{X})$ and is closed in the uniform operator topology.

If $\mathcal{F}(\mathfrak{X})$ is the set of operators with finite rank, then $\mathcal{F}(\mathfrak{X})$ is a 2-sided ideal, but it is not necessarily closed.

DEFINITION: A B-space $\mathcal{X}$ has the approximation property if for any B-space $\mathcal{Z}$ and any $T \in \mathcal{K}(\mathcal{Z}, \mathcal{X})$ there exists a sequence of finite rank operators F_n with

$$\| F_n - T \| \longrightarrow 0$$

1973 Per Enflo (Acta Math) $\exists\, \mathcal{X}$ without the approx. property

7.1.7. THEOREM (Schauder) Let $\mathcal{X}, \mathcal{Y}$ be B-spaces, $T \in B(\mathcal{X}, \mathcal{Y})$. Then T is compact iff T^* is compact.

Proof. a) Suppose T is compact. Let $\Omega = \overline{T(S)} \subset \mathcal{Y}$, so Ω is compact. Let (y_n^*) be a bounded seq. in $\mathcal{Y}^*$, say

$$\| y_n^* \| \leq M$$

Regard $y_n^* \in C(\Omega)$. Then

$$| y_n^*(y_1) - y_n^*(y_2) | \leq \| y_n^* \| \| y_1 - y_2 \| \leq M \| y_1 - y_2 \|$$

for all $y_1, y_2 \in \Omega$, and so (y_n^*) is uniformly equicontinuous set in $C(\Omega)$. Also, (y_n^*) is uniformly bounded in $C(\Omega)$. Hence, by the Arzela-Ascoli theorem, there exists a subseq. $(y_{n_k}^*)$ which is convergent in $C(\Omega)$, i.e.

$$y_n^* \longrightarrow \xi \quad \text{unif. on } \Omega = \overline{T(S)}$$

Therefore $(T^* y_n^*)$ converges unif on S.

$$\forall x \in S, \quad |T^* y_{n_k}^*(x) - T^* y_{n_\ell}^*(x)|$$

$$= |y_{n_k}^*(Tx) - y_{n_\ell}^*(Tx)| \longrightarrow 0$$

$$\Rightarrow \|T^* y_{n_k}^* - T^* y_{n_\ell}^*\| \to 0 \Rightarrow T^* \text{ compact}$$

Conversely, suppose T^* is compact. Then $T^{**} : \mathcal{X}^{**} \to \mathcal{Y}^{**}$ is compact. But

$$T = T^{**}\Big|_{\kappa(\mathcal{X})}$$

$\uparrow$ canonical embedding

Hence T is compact.

$\blacksquare$

7.1.8. <u>THEOREM:</u> Let $\mathcal{X}$ be reflexive. Then $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is compact if and only if T maps weakly convergent sequences in $\mathcal{X}$ into norm convergent sequence in $\mathcal{Y}$

7.1.9. <u>THEOREM:</u> If $\mathfrak{H}$ is a Hilbert space, then $T \in \mathcal{B}(\mathfrak{H})$ is compact if and only if $\exists\ (F_n)$ of finite rank operators with

$$\|F_n - T\| \longrightarrow 0$$

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<u>COROLLARY:</u> $T \in B(\mathcal{H})$ compact $\Leftrightarrow \|F_n - T\| \to 0$ for F_n finite rank

(Use $1/n$-nets and projections onto subspaces)

§7.2 SPECTRAL THEORY OF COMPACT OPERATORS

7.2.1. <u>Riesz's Lemma</u>: Let $\mathfrak{X}$ be a normed linear space, $\mathfrak{X}_0$ a proper closed subspace of $\mathfrak{X}$. If $0 < \varepsilon < 1$ there exists x_0 with $\|x_0\| = 1$ and $d(x_0, \mathfrak{X}_0) \geq 1 - \varepsilon$.

Proof. Take $y \in \mathfrak{X} \setminus \mathfrak{X}_0$. Let $d := \text{dist}(y, \mathfrak{X}_0) > 0$. Then there exists $u \in \mathfrak{X}_0$ with

$$0 < \|y - u\| < \frac{d}{1-\varepsilon}$$

Let $x_0 = y-u / \|y-u\|$. Then $\|x_0\| = 1$. If $w \in \mathfrak{X}_0$, then

$$\|x_0 - w\| = \left\| \frac{y - u - w\|y-u\|}{\|y-u\|} \right\| = \frac{1}{\|y-u\|} \|y - u - w\|y-u\|\|$$

$$= \frac{1}{\|y-u\|} \|y - \underbrace{(u + w\|y-u\|)}_{\in \mathfrak{X}_0}\| \geq \frac{d}{\|y-u\|} > 1 - \varepsilon$$

$\blacksquare$

(THEOREM: Let $\mathcal{M}, \mathcal{N}$ be subspaces of $\mathcal{X}$ with $\dim \mathcal{M} > \dim \mathcal{N}$. Then there exists $x \in \mathcal{M}$ s.t.

$$d(x, \mathcal{N}) = \|x\| > 0$$)

7.2.2. <u>COROLLARY</u>: Let $\mathcal{X}$ be a normed linear space, S unit ball of $\mathcal{X}$. TFAE

(1) $\mathcal{X}$ is finite dimensional

(2) S is compact

(3) S is totally bounded

Proof (3) $\Rightarrow$ (1). If $\mathcal{X}$ is not finite dimensional, let $\{x_1, x_2, \dots\}$ be an infinite linearly independent set. Using Riesz's lemma, we can choose x_n s.t. $\|x_n\| = 1$ and $\|x_n - x_m\| > 1/2$ $\forall n \neq m$. Then S is not totally bounded

▨

7.2.3. <u>LEMMA</u>: Let $T \in \mathcal{K}(\mathcal{X})$. If $\lambda \neq 0$, then either $\lambda \in \sigma_p(T)$ or $R(\lambda I - T)$ is closed.

7.2.4. <u>COROLLARY</u>: T compact, $\lambda \neq 0 \Rightarrow \lambda \notin \sigma_c(T)$.

Proof of 7.2.3. Let $y_n = (\lambda I - T)x_n$, $y_n \to y_0$

Case i : (x_n) has a bdd subsequence. $\exists$ subseq. x_{n_k} s.t. $Tx_{n_k} \to$ some z_0.

$$\begin{array}{ccc} y_{n_k} & = & \lambda x_{n_k} - Tx_{n_k} \\ \downarrow & & \downarrow \\ y_0 & & z_0 \end{array}$$

$$\Rightarrow x_{n_k} \to \frac{1}{\lambda}(y_0 + z_0) =: x_0$$

$$\Rightarrow y_0 = \lim (\lambda I - T)x_{n_k} = (\lambda I - T)x_0$$

so $R(\lambda I - T)$ is closed.

Case ii : (x_n) has no bounded subsequence. Then $\|x_n\| \to \infty$.
Let

$$z_n := \frac{x_n}{\|x_n\|}$$

$$\Rightarrow (\lambda - T)z_n = (\lambda - T)\frac{x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to 0$$

Since T is compact $Tz_{n_k} \to w$ for some subsequence. Hence

$$\lambda z_{n_k} - \underset{\substack{\downarrow \\ w}}{Tz_{n_k}} \longrightarrow 0$$

$$\Rightarrow \quad z_{n_k} \longrightarrow \frac{1}{\lambda} w \neq 0$$

$$\Rightarrow \quad (\lambda - T)(z_{n_k}) \longrightarrow 0$$
$$\downarrow$$
$$(\lambda - T)\left(\frac{1}{\lambda} w\right)$$

$$\Rightarrow \quad (\lambda - T)(w) = 0$$

Hence $\lambda \in \sigma_p(T)$. $\boxtimes$

7.2.5. <u>LEMMA:</u> T compact. Then $\sigma_p(T)$ has no non-zero cluster point in $\mathbb{C}$.

Proof. If $\mathcal{X}$ is finite dimensional, σ is finite ✓.
Suppose there exist a sequence (λ_n) of distinct elements in $\sigma_p(T)$ s.t. $\lambda_n \to \lambda \neq 0$. $\exists\, x_n \neq 0$ s.t. $Tx_n = \lambda_n x_n$.

CLAIM: For $n \in \mathbb{N}$, the set $\{x_1, \dots, x_n\}$ is linearly independent.
Suppose $\{x_1, \dots, x_k\}$ is linearly independent, but $\{x_1, \dots, x_{k+1}\}$ is linearly dependent. Then

$$x_{k+1} = \alpha_1 x_1 + \dots + \alpha_k x_k$$

$$\Rightarrow \quad T x_{k+1} = \alpha_1 T x_1 + \dots + \alpha_k T x_k$$

$$\Rightarrow \quad x_{k+1} = \alpha_1 \frac{\lambda_1}{\lambda_{k+1}} x_1 + \dots + \alpha_k \frac{\lambda_k}{\lambda_{k+1}} x_n$$

$$\Rightarrow \quad 0 = \alpha_1 \underbrace{\left(1 - \frac{\lambda_1}{\lambda_{n+1}}\right)}_{\neq 0} x_1 + \dots + \alpha_k \underbrace{\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)}_{\neq 0} x_k$$

Let $\mathcal{Z}_n = \mathrm{sp}\{x_1, \dots, x_n\}$. Then $\mathcal{Z}_n$ is closed in $\mathcal{X}$ and $\mathcal{Z}_n \subsetneq \mathcal{Z}_{n+1}$. By Riesz, $\exists\, y_{m+1} \in \mathcal{Z}_{m+1}$ s.t. $\|y_{m+1}\| = 1$ and

$$\| y_{m+1} - x \| > 1/2$$

$\forall x \in \mathcal{Z}_m$. Then

$$y_{m+1} = \gamma_1 x_1 + \dots + \gamma_{m+1} x_{m+1}$$

where $\gamma_{m+1} \neq 0$.

$$(\lambda_{m+1} - T) y_{m+1} = \underbrace{(\lambda_{m+1} - T)(\gamma_1 x_1 + \dots + \gamma_m x_m)}_{\in \mathcal{Z}_m} + 0$$

from x_{m+1}

$$\Rightarrow \quad T y_{m+1} = \underbrace{\lambda_{m+1} y_{m+1}}_{\in \mathcal{Z}_{m+1}} + \underbrace{z_m}_{\in \mathcal{Z}_m}$$

Hence if $m > n$

$$\| T y_m - T y_n \| = \| \lambda_m y_m + \underbrace{\tilde{z}_{m-1}}_{\in \mathcal{Z}_{m-1}} \|$$

$$= |\lambda_m| \, \| y_m + \tilde{\tilde{z}}_{m-1} \| \geq \frac{1}{2} |\lambda_m| > \varepsilon$$

for sufficiently large m

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7.2.6. LEMMA: Let $T \in \mathcal{K}(\mathfrak{X})$ and let $\lambda \neq 0$, $\lambda \in \sigma_a(T)$. Then $\lambda \in \sigma_p(T)$

Proof. $\exists\ (x_n)$ $\|x_n\|=1$ s.t. $y_n := (\lambda - T)x_n \to 0$.

$$x_n = \frac{1}{\lambda}(Tx_n + y_n)$$

$\exists$ subseq x_{n_k} s.t. $Tx_{n_k} \to z$, so $x_{n_k} = \frac{1}{\lambda}(Tx_{n_k} + y_{n_k}) \to \frac{1}{\lambda} z$
Hence $\|z\| \neq 0$ since $\|x_{n_k}\|=1$, and

$$\begin{array}{ccc} Tx_{n_k} & \longrightarrow & \frac{1}{\lambda} Tz \\ \downarrow & & \\ z & & \end{array}$$

$$\Rightarrow Tz = \lambda z$$

Hence λ is an eigenvalue. ▨

7.27. THEOREM: Let $T \in \mathcal{K}(\mathfrak{X})$. Then $\sigma(T)$ is countable with at most $\lambda = 0$ as a cluster point. If $\lambda \in \sigma(T)$, $\lambda \neq 0$, then $\lambda \in \sigma_p(T)$ and the eigenspace corresponding to λ is finite dimensional. In fact $\lambda \in \sigma(T)$, $\lambda \neq 0$, is a pole of $R(\lambda; T)$ and

$$E(\lambda; T) = \frac{1}{2\pi i} \int_{C(\lambda)} R(\lambda; T)\, d\lambda$$

has finite dimensional range.

also compact

Proof. $\sigma_p(T)$ has no non-zero cluster point. $\sigma_r(T) \subset \sigma_p(T^*)$ also has no non-zero cluster point. Hence $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \{0\}$ has no non-zero cluster point.

$$\partial\sigma(T) = \sigma(T) \subset \sigma_p(T) \cup \{0\}$$

(If $\mathfrak{X}$ is infinite dimensional, then $0 \in \sigma(T)$ since if $\exists$ only a finite number of λ_j, then $I = E(\lambda_1;T) + \dots + E(\lambda_n;T)$)

Claim: $\mathcal{N}(\lambda_0 I - T)$ is finite dimensional for $\lambda_0 \neq 0$, $\lambda_0 \in \sigma(T)$.

Suppose $\{x_1, \dots, x_n, \dots\}$ are eigenvectors corresponding to λ_0. Suppose this set is linearly independent. Let $Z_n = sp\{x_1, \dots, x_n\}$. Then $Z_n \subsetneq Z_{n+1}$. Choose $z_n \in Z_n$ with $\|z_n\| = 1$ and $dist(z_n, Z_{n-1}) > 1/2$. Then

$$m > n \implies \|z_m - z_n\| > 1/2$$

$$\implies \|Tz_m - Tz_n\| = |\lambda_0| \|z_m - z_n\| > 1/2 |\lambda_0|$$

If $\lambda \in \rho(T)$, $\lambda \neq 0$, let $S(\lambda) := R(\lambda;T) - \frac{1}{\lambda} I$. Then

$$I = (\lambda - T)R(\lambda) = \lambda S(\lambda) - TS(\lambda) + I - \frac{1}{\lambda}T$$

$$\implies \lambda S(\lambda) = TS(\lambda) + \frac{1}{\lambda}T$$

$$\implies S(\lambda) = T\left(\frac{1}{\lambda}S(\lambda) + \frac{1}{\lambda^2}I\right) \in \mathcal{K}(\mathfrak{X})$$

If $\lambda_0 \in \sigma(T), \lambda_0 \neq 0$, take a contour such that the inside of C intersects $\sigma(T)$ only at λ_0 (possible since λ_0 is isolated).

$$E(\lambda_0;T) = \frac{1}{2\pi i}\int_C R(\lambda;T)\,d\lambda$$

analytic since $0 \notin$ inside C

$$= \frac{1}{2\pi i}\int_C \left(S(\lambda) + \frac{1}{\lambda}I\right) d\lambda$$

limit of compact operators

$$= \frac{1}{2\pi i}\int_C S(\lambda) \in \mathcal{K}(\mathfrak{X})$$

Since the projection $E(\lambda_0;T)$ is compact, it has finite dimensional range.

$\boxtimes$

Corollary: $T \in \mathcal{K}(\mathfrak{X})$. If $\lambda \neq 0$, then $\mathcal{R}(\lambda - T)$ is closed

Proof. Either $\lambda \in \sigma_p(T)$ or $\lambda \notin \sigma(T)$. In the latter case $\mathcal{R}(\lambda - T) = \mathfrak{X}$ is clearly closed. If $\lambda \in \sigma_p(T)$, then

$$\mathfrak{X} = \underbrace{E(\lambda)\mathfrak{X}}_{\text{finite dim}} \oplus E(\sigma(T)\setminus\{\lambda\})\mathfrak{X}$$

$$(\lambda - T)\mathfrak{X} = \underbrace{(\lambda - T)E(\lambda)\mathfrak{X}}_{\substack{\text{finite dim} \\ \Rightarrow \text{closed range}}} \oplus \underbrace{(\lambda - T)E(\lambda')\mathfrak{X}}_{\text{invertible} \Rightarrow \text{closed range}}$$

Hence $(\lambda - T)\mathfrak{X}$ is closed

$\boxtimes$

7.2.8 Fredholm Alternative: $T \in \mathcal{K}(\mathfrak{X})$, $\lambda \neq 0$. Then the operator $\lambda I - T$ is one-one if and only if $\lambda I - T$ is onto. Either

$$(E) \qquad y = (\lambda - T)x$$

has a unique solution for all $y \in \mathfrak{X}$, or the homogeneous equation

$$(H) \qquad 0 = (\lambda - T)x$$

has finitely many, linearly ind non-trivial solutions. In the first case

$$(E^*) \qquad y^* = (\lambda I^* - T^*)x^*$$

has a unique solution for all $y^* \in \mathfrak{X}^*$. In the second case

$$(H^*) \qquad 0 = (\lambda I^* - T^*)x^*$$

has the same number of linear independent non-trivial solutions. Moreover, in the second case, (E) has a solution for y iff $x^* y = 0$ $\forall x^*$ satisfying (H^*), and (E^*) has a solution for y^* iff $y^* x = 0$ $\forall x$ satisfying H.

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7.2.9. <u>THEOREM:</u> Let T be a compact normal operator in $B(\mathcal{H})$. Let $\{\lambda_j\}$ be the sequence of distinct eigenvalues of T arranged so that

$$\underset{\text{if necessary}}{\lambda_0 = 0}, \quad |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| \geq \ldots$$

$(\lambda_i \neq \lambda_j \text{ if } i \neq j)$ Let E_j be the orthogonal projection of $\mathcal{H}$ onto the jth eigenspace. Let $\eta_j = \eta(\lambda_j I - T)$. Then E_j has finite rank and are mutually orthogonal. Moreover,

$$I = \sum_{i=0}^{\infty} E_i \quad \text{(strong op. top)}$$

$$T = \sum_{i=0}^{\infty} \lambda_i E_i \quad \text{(uniform operator top)}$$

In particular

$$\left\| T - \sum_{i=0}^{n} \lambda_i E_i \right\| = |\lambda_{n+1}|$$

Proof. Since T is normal, all root vectors have height 1. Also, eigenvectors corresponding to distinct eigenvalues are orthogonal. Let

$$e_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n, 0\} \qquad e_n \uparrow \sigma(T)$$

Since $E_j = E(\{\lambda_j\})$ and E is strongly countably additive,

$$I = E(\sigma(T)) = \lim_{n \to \infty} E(e_n) \quad \text{in strong op. top}$$

[$(\lambda - T)x = 0$ and $(\mu - T)y = 0$, then $(\mu^* - T^*)y = 0$, so

$$\lambda\langle x,y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \mu^* y\rangle = \mu\langle x,y\rangle$$

so if $\lambda \neq \mu$ then $\langle x,y\rangle = 0$]

Let $S_n = T - \sum_{i=1}^{n} \lambda_i E_i$. This is normal. Then

$$\sigma(S_n) = \sigma(T) - \{\lambda_1, \ldots, \lambda_n\}$$

and $\|S_n\| = r(S_n) = |\lambda_{n+1}|$ ▨

7.2.10 <u>THEOREM:</u> Let $T \in B(\mathcal{H})$ be compact and normal. Then there exist a complete orthonormal set $\{x_\alpha\}_{\alpha \in A}$ of eigenvectors for T. Hence

$$x = \sum_{\alpha \in A} \langle x, x_\alpha\rangle x_\alpha$$

$$Tx = \sum_{\alpha \in A} \lambda_\alpha \langle x, x_\alpha\rangle x_\alpha$$

(really just a countably inf. series)

Proof. Pick an orthonormal basis in $\mathcal{N}(\lambda_i I - T)$ for each $\lambda_i \in \sigma_p(T)$. Now

$$I = \sum_{j \in \sigma_p(T)} E(\lambda_j) \quad \text{(strong op. top)}$$

7.2.11 **THEOREM:** Let $T \in B(\mathcal{X})$ such that for some $n \in \mathbb{N}$, $T^n \in \mathcal{K}(\mathcal{X})$. Then the non-zero spectrum of T is countable with $\lambda = 0$ as the only possible cluster point. Each $\lambda \neq 0$, $\lambda \in \sigma(T)$ is an eigenvalue with finite multiplicity and $E(\lambda)$ has finite dimensional range.

Proof. Let $T^n = K \in \mathcal{K}(\mathcal{X})$. Then $\sigma(K) = \sigma(T^n) = (\sigma(T))^n$
Hence $\sigma(T) \subset (\sigma(K))^{1/n}$ spectral mapping theorem

all complex nth roots

Result follows from corresponding conclusions for K

Let $\lambda \in \sigma(T)$. Then $\lambda^n \in \sigma(K)$. Let $\lambda_1, \ldots, \lambda_r$ be the points in $\sigma(T)$ such that

$$\lambda_1^n = \ldots = \lambda_r^n = \lambda^n$$

Then $\{\lambda_1, \ldots, \lambda_r\}$ is clopen in $\sigma(T)$ [[since discrete]]. Let $\tau = \{\lambda^n\}$ Then

$$\{\lambda_1, \ldots, \lambda_r\} = \sigma(T) \cap f^{-1}(\tau)$$

where $f = \iota^n$. By the preceding theorem

$$E(\lambda^n; T^n = K) = E(\{\lambda_1, \ldots, \lambda_r\}; T) = \sum_{i=1}^{r} E(\lambda_i; T)$$

$$\parallel$$

$$E(\{\lambda^n\}; K)$$

finite dimensional $\Rightarrow \sum_{i=1}^{r} E(\lambda_i; T)$ is finite dimensional

$\boxed{}$

Weakly Compact Operators

7.3.12 **DEFINITION:** $T \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is weakly compact if the weak closure of $T(\overline{B})$ is weakly compact in $\mathcal{Y}$

By Eberlein-Smulian theorem, T is weakly compact iff for every bounded seq. (x_n) in $\mathcal{X}$, (Tx_n) has a weakly convergent subsequence.

$\mathcal{W}(\mathcal{X},\mathcal{Y})$ = set of all weakly compact operators

7.3.13 **THEOREM:** (DS VI.4.2) $T \in \mathcal{W}(\mathcal{X},\mathcal{Y}) \Leftrightarrow T^{**}\mathcal{X}^{**} \subset Q\mathcal{Y}$

7.3.14 **COROLLARY:** If either $\mathcal{X}$ or $\mathcal{Y}$ is reflexive, then $\mathcal{W}(\mathcal{X},\mathcal{Y}) = \mathcal{B}(\mathcal{X},\mathcal{Y})$.

7.3.15 **COROLLARY:** $\mathcal{W}(\mathcal{X},\mathcal{Y})$ is a normed closed subspace of $\mathcal{B}(\mathcal{X},\mathcal{Y})$
The product of a weakly compact operator with a bounded operator is weakly.

7.3.16 **COROLLARY:** The set $\mathcal{W}(\mathcal{X})$ is a normed close two-sided ideal in $\mathcal{B}(\mathcal{X})$.

7.3.17 **THEOREM:** $T \in \mathcal{W}(\mathcal{X},\mathcal{Y})$ iff $T^*: \mathcal{Y}^* \to \mathcal{X}^*$ is w^*-w cont

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LEMMA: Let $T \in B(\mathfrak{H})$ be compact and normal. Then the set of eigenvectors for T is complete (i.e. $z \perp \eta(\lambda - T)\ \forall \lambda \in \mathbb{C} \Rightarrow z = 0$)

Proof. $\eta(\lambda - T) = \eta(\lambda^* - T^*)$ since $\|(\lambda - T)x\| = \|(\lambda^* - T^*)x\|$. Let

$$\mathfrak{M} = \overline{\text{span}}\{\eta(\lambda - T) : \lambda \in \mathbb{C}\}$$

Let $\eta = \mathfrak{M}^\perp$. Then $\mathfrak{M}, \eta$ reduce T, T^*. Let $T_1 := T|\eta$. T_1 is normal and compact. If $\mu \neq 0$ and $\mu \in \sigma(T_1) = \sigma_p(T_1) \subset \sigma_p(T)$, then η contains an eigenvector for T corresponding to μ. Hence $\sigma(T_1) \subset \{0\}$. If $\sigma(T_1) = \{0\}$, then since T_1 is normal, $\|T_1\| = r(T_1) = 0$, so $T_1 = 0$. Thus $\lambda = 0$ is an eigenvalue for T ↯. Hence $\sigma(T_1) = \emptyset \Rightarrow \eta = \{0\}$

▨

THEOREM: (DS VII, 3.19) Let $f \in \mathcal{F}(T)$ and let τ be clopen in $\sigma(f(T)) = f(\sigma(T))$. Then $\sigma(T) \cap f^{-1}(\tau)$ is clopen in $\sigma(T)$ and

$$E(\tau; f(T)) = E(f^{-1}(\tau); T)$$

(If $0 \in \sigma_c(T)$, then $E(\{0\}) = 0$). Hence

$$X = \sum_{\alpha \in A} \langle x, x_\alpha \rangle x_\alpha$$

Since $E(\lambda_j)x = \sum \langle x, x_\alpha \rangle x_\alpha$ where $\{x_\alpha : \alpha \in B_j\}$ is basis for $\mathcal{N}(\lambda_j - T) = E(\lambda_j)\mathcal{H}$

$\blacksquare$

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7.2.17 THEOREM: $T \in B(\mathfrak{X},\mathcal{Y})$ is weakly compact iff T is weak* to weak continuous

7.2.18 THEOREM (Gantmacher) $T \in \mathcal{W}(\mathfrak{X},\mathcal{Y})$ iff $T^* \in \mathcal{W}(\mathcal{Y}^*,\mathfrak{X}^*)$

7.2.19 THEOREM (DS VI.7.3) Let K be a compact Hausdorff space. Then $T: C(K) \to \mathfrak{X}$ is weakly compact iff there exists a regular strongly countably additive measure μ on the Borel sets of K into $\mathfrak{X}$ such that

$$Tf = \int_K f \, d\mu$$

7.2.20 THEOREM (DS VI.7.4) If $T: C(K) \to \mathfrak{X}$ is weakly compact, then T maps weakly Cauchy sequences in $C(K)$ into strongly convergent sequences in $\mathfrak{X}$. Hence T maps conditionally weakly compact sets in $C(K)$ into relatively norm compact sets in $\mathfrak{X}$. [i.e. T is Dunford-Pettis]

7.2.21 COROLLARY: The product of two weakly compact operators in $C(K)$ is strongly compact

If $T: C(K) \to$ weakly complete space (e.g. L_1) is weakly compact.

7.2.22 <u>THEOREM</u> (DS VI.8.10) Let (Ω, Σ, μ) be σ-finite and let $\mathfrak{X}$ be separable. If $T: L_1(\mu) \to \mathfrak{X}$, then there exist a bounded measurable $g: \Omega \to \mathfrak{X}$ with weakly compact range s.t

$$Tf = \int_\Omega f g \, d\mu$$

7.2.23 <u>THEOREM:</u> (DS VI, 8.12) Let (Ω, Σ, μ) be a positive measure space. If $T: L_1(\mu) \to \mathfrak{X}$ is weakly compact, then T maps weakly Cauchy seq. in L_1 into norm convergent subsequence in $\mathfrak{X}$.

7.2.24 <u>COROLLARY:</u> The product of two weakly compact operators in $L_1(\Omega, \Sigma, \mu)$ is strongly compact.

7.2.25 <u>DEFINITION:</u> An operator $T \in B(\mathfrak{X}, \mathcal{Y})$ is strictly singular if it does not have a bounded inverse on any infinite dimensional subspace of $\mathfrak{X}$.

(Denote by $\mathcal{S}(\mathfrak{X}, \mathcal{Y})$)

[T. Kato J. d'Analyse Math 6 (1958), p261-322]
[Seymour Goldberg "Unbounded linear operators and Applications" McGraw Hill, 1966]

7.2.26 THEOREM: $\mathcal{K}(\mathcal{X},\mathcal{Y}) \subset \mathcal{S}(\mathcal{X},\mathcal{Y})$

Proof. Let $\mathcal{M} \subset \mathcal{X}$ be a subspace on which $T \in \mathcal{K}(\mathcal{X},\mathcal{Y})$ is invertible. If $S_{\mathcal{M}}$ is the unit ball of $\mathcal{M}$, then $TS_{\mathcal{M}}$ is totally bounded. Hence

$$S_{\mathcal{M}} = \underset{\text{cont.}}{\underset{\uparrow}{(T|_{\mathcal{M}})^{-1}}} \; \underset{\text{total bdd}}{\underset{\uparrow}{(TS_{\mathcal{M}})}}$$

$\Rightarrow S_{\mathcal{M}}$ totally bdd

$\Rightarrow \mathcal{M}$ finite dimensional

Hence T is strictly singular.

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7.2.27 THEOREM Let $S \in B(\mathcal{X}, \mathcal{Y})$. TFAE

(i) S is strictly singular

(ii) For any infinite dimensional subspace $\mathcal{M} \subset \mathcal{X}$ there exist an infinite dimensional $\mathcal{N} \subset \mathcal{M}$ such that $S|_{\mathcal{N}}$ is compact.

(iii) Given $\varepsilon > 0$ and an infinite dimensional $\mathcal{M} \subset \mathcal{X}$, there exist an infinite dimensional $\mathcal{N} \subset \mathcal{M}$ s.t. $\|S|_{\mathcal{N}}\| < \varepsilon$

7.2.28 THEOREM (Whitley) A weakly compact operator that maps weakly convergent sequences into norm convergent seq. is strictly singular.

Proof. Let $\mathcal{M} \subset \mathcal{X}$ be a closed subspace on which T is invertible. Let $(x_n) \subset \mathcal{M}$ be a bounded seq. Then $\exists\ (x_{n_k})$ s.t. $Tx_{n_k} \to y_0$ weakly So $y_0 \in T\mathcal{M}$. Since $T|_m$ is invertible, the seq (x_{n_k}) converges weakly to $(T|_m)^{-1} y_0$, and so

$$Tx_{n_k} \longrightarrow T(T|_m)^{-1} y_0 = y_0$$

in norm. Hence $x_{n_k} \to (T|_m)^{-1} y_0$ in norm. We have shown that any closed bounded set is compact, so $\mathcal{M}$ is finite dimensional.

$\blacksquare$

7.2.29 <u>COROLLARY:</u> Let $\mathfrak{X} = L_1$ or $\mathfrak{X} = C(K)$. Then $\mathcal{W}(\mathfrak{X}) \subset \mathcal{S}(\mathfrak{X})$
Every bounded on $C(K)$ into a weakly complete space is strictly singular.

7.2.30 <u>THEOREM:</u> (See Goldberg p86)

(a) The set $\mathcal{S}(\mathfrak{X}, \mathcal{Y})$ is closed in $B(\mathfrak{X})$

(b) The product of a strictly singular operator and a bounded operator is strictly singular.

(c) The set $\mathcal{S}(\mathfrak{X})$ is a closed 2-sided ideal in $B(\mathfrak{X})$.

J.W. Calkin [Ann. of Math 42 (1941)] : $\mathcal{K}(\ell_2)$ is the only closed two sided ideal in $B(\ell_2)$

I C. Goldberg, A Markus, I. Fel'dman [AMS Transl (2) #61 (1967)] : $\mathcal{K}(\ell_p)$ and $\mathcal{K}(c_0)$ are the only two sided ideals in $B(\ell_p)$, $B(c_0)$ respectively
$1 < p < \infty$

A. Pelcyznski [Bull. Acad. Polon 13 (1965)] : $\mathcal{W} = \mathcal{S}$ in $B(C(K), \mathcal{Y})$ or $B(L_1)$

H. Porta [BAMS 75 (1969)]

§7.3 Nuclear and Related Operators

Let $\mathcal{X}, \mathcal{Y}$ be B-spaces. If $x^* \in \mathcal{X}$, $y \in \mathcal{Y}$, let $x^* \otimes y \in B(\mathcal{X}, \mathcal{Y})$ be given by

$$(x^* \otimes y)x := x^*(x)y$$

7.3.1. LEMMA: $T \in B(\mathcal{X}, \mathcal{Y})$ has finite dimensional range, if and only if

$$T = \sum_{j=1}^{n} x_j^* \otimes y_j$$

Proof ($\Leftarrow$) clear

($\Rightarrow$) Let $\{y_1, \dots, y_n\}$ be a basis for the range of T. Let $\{y_1^*, \dots, y_n^*\}$ be the dual basis. Let

$$x_j^* = T^* y_j^*$$

Then

$$Tx = \sum x_j^*(x) y_j = \sum_{j=1}^{n} x_j^* \otimes y_j$$

7.3.2. LEMMA: If (x_j^*) (y_j) are sequences s.t

$$\sum \|x_j^*\| \, \|y_j\| < \infty$$

Then for each $x \in \mathcal{X}$,

$$\sum_{j=1}^{\infty} x_j^*(x) \, y_j \quad \text{converges absolutely in } \mathcal{Y}$$

If we define

$$Tx = \sum \left(x_j^* x \right) y_j$$

Then $T \in B(\mathcal{X})$ and $\|T\| \leq M$.

Proof: $\| \sum (x_j^* x) y_j \| \leq \left(\sum \|x_j^*\| \|y_j\| \right) \|x\| \leq M \|x\|$

7.3.3. DEFINITION: If $T \in B(\mathcal{X}, \mathcal{Y})$ is such that there exist sequences $(x_j^*) \subset \mathcal{X}^*$ and $y_i \in \mathcal{Y}$ s.t $\sum \|x_j^*\| \|y_j\| < \infty$ and

$$Tx = \sum x_j^*(x) \, y_j$$

Then T is a nuclear operator and we write

$$T = \sum x_j^* \otimes y_j$$

[$\eta(X, Y)$ = all nuclear maps of $X \to Y$]

7.3.4. LEMMA: If $T \in \mathcal{N}(\mathcal{X}, \mathcal{Y})$ we can write T in the form

$$T = \sum_{j=1}^{\infty} \alpha_i \, z_i^* \otimes w_j$$

where $z_i^* \in \mathcal{X}^*$, $y \in \mathcal{Y}$ and $\|z_j^*\| = 1 = \|w_i\|$

Proof Let $T \in \sum x_j^* \otimes y_j$. Let $\alpha_j = \|x_j\| \|y_j\|$
If $\alpha \neq 0$, let

$$z_i^x = \frac{x_i^*}{\|x_j\|^2}$$

$$w_i = \frac{y_j}{\|y_j\|}$$

If $\alpha_1 = 0$, drop. In either case,

$$\alpha_i \, z_i^* \otimes x_\gamma^{\perp} \otimes y,$$

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$T \in \mathcal{N}(\mathfrak{X}, \mathcal{Y})$ if $T = \sum_{j=1}^{\infty} x_j^* \otimes y_j$ where $\sum \|x_j^*\| \|y_j\| < \infty$

[[equivalently $T = \sum \alpha_j (z_j^* \otimes w_j)$ where $\|z_j^*\| = 1 = \|w_j\|$, $\sum |\alpha_j| < \infty$]]

7.3.5 THEOREM:

(a) $\mathcal{N}(\mathfrak{X}, \mathcal{Y})$ is a vector space

(b) Every finite rank operator is nuclear

(c) Every nuclear operator is compact

Proof: (c) $$\Big\| Tx - \underbrace{\sum_{j=1}^{n} \alpha_j (z_j^* x) w_j}_{\text{finite rank operator}} \Big\| \le \Big(\sum_{j=n+1}^{\infty} |\alpha_j| \Big) \|x\| \to 0$$
(unif in $\|x\| \le 1$)

$\Rightarrow T$ compact

$\boxtimes$

Consider case when $\mathfrak{X} = \mathfrak{H}_1$, $\mathcal{Y} = \mathfrak{H}_2$. Define

$$(e \otimes f)(x) := \langle x, e \rangle f$$

7.3.6 THEOREM Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces. Let $(e_n) \subset \mathfrak{H}_1$, $(f_n) \subset \mathfrak{H}_2$ be orthonormal sequences. Let (α_n) be a bounded sequence in $\mathbb{C}$. Let

$$T_n = \sum_{i=1}^{n} \alpha_i (e_i \otimes f_i)$$

(a) There exists $T \in B(\mathfrak{H}_1, \mathfrak{H}_2)$ such that $T_n x \to Tx$ for every $x \in \mathfrak{H}_1$

(b) If $\alpha_n \to 0$, then $\|T_n - T\| \to 0$ and T is compact.

Proof. (a) Let $x \in \mathfrak{H}_1$, $m > n$. Then

$$\|T_m x - T_n x\|^2 = \left\langle \sum_{n+1}^{m} \alpha_i \langle x, e_i \rangle f_i , \sum_{n+1}^{m} \alpha_j \langle x, e_j \rangle f_j \right\rangle$$

$$= \sum_{n+1}^{m} |\alpha_i|^2 |\langle x, e_i \rangle|^2$$

$$\leq \left(\sup_{i>n} |\alpha_i|^2 \right) \sum_{n+1}^{m} |\langle x, e_i \rangle|^2$$

$$\leq \left(\sup_{i>n} |\alpha_i|^2 \right) \varepsilon \quad \text{for sufficiently large } n$$

Therefore $(T_n x)$ is a Cauchy sequence in $\mathfrak{H}_2$. Let $Tx := \lim T_n x$

(b) By the same calculation as in (a),

$$\|Tx - T_n x\|^2 \leq \left(\sup_{i>n} |\alpha_i|^2 \right) \|x\|^2 \quad \text{(using Bessel's ineq.)}$$

and so if $\alpha_j \to 0$, then $\| T_n - T \| \to 0$.

$\blacksquare$

7.3.7 Example. If $\mathfrak{H}$ is separable and if (e_n) is an orthonormal basis for $\mathfrak{H}$, then

$$I = \sum_{n=1}^{\infty} e_n \otimes e_n$$

(in sense of (a)).

Proof. Let $P_n = \sum_{i=1}^{n} e_i \otimes e_i$. Then P_n is the orthonormal projection of $\mathfrak{H}$ onto $\overline{\text{span}}\,(e_1, \ldots, e_n)$

[[Note: if $T = \sum \alpha_i \, e_i \otimes f_i$, then $T^* = \sum \alpha_i^* \, f_i \otimes e_i$]]

$$Ix = \sum \langle x, e_i \rangle e_i$$

$\blacksquare$

7.3.8. THEOREM: Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces, $T \in B(\mathfrak{H}_1, \mathfrak{H}_2)$. Then T is compact iff it can be represented in the form

$$T = \sum_{i=1}^{\infty} \alpha_i (e_i \otimes f_i)$$

where $(e_i), (f_i)$ are orthonormal sequences and $\alpha_n \to 0$.

Proof. ($\Rightarrow$) Suppose $H: \mathfrak{h} \to \mathfrak{h}$ is compact and $H^* = H$. Then

$$H = \sum_{i=1}^{\infty} \lambda_i P_i$$

where (λ_i) is the sequence of non-zero eigenvalues of H and P_i is the orthonormal projection onto this eigenspace. If $(c_j : j = 1, \dots, n_i)$ is a basis for the eigenspace for λ_i, then

$$P_i = \sum_{j=1}^{n_i} c_j \otimes c_j$$

Now relabel to get

$$H = \sum \underbrace{\alpha_i}_{\text{eigenvalues counted by multiplicity}} (e_i \otimes e_i)$$

Now consider a general compact operator $T: \mathfrak{h}_1 \to \mathfrak{h}_2$. Use polar representation (canonical factorization). Let $T^*: \mathfrak{h}_2 \to \mathfrak{h}_1$ satisfy $\langle Tx, y \rangle_2 = \langle x, T^* y \rangle_1$. Let $H_1 = T^*T : \mathfrak{h}_1 \to \mathfrak{h}_1$.

Then H_1 is Hermitian and compact. Let $H = \sqrt{H_1}$. This is Hermitian and compact

Define the partial isometry $U: \overline{\mathcal{R}(H)} \to \mathcal{R}(T)$ by $T = UH$

$$H = \sum \alpha_i (e_i \otimes e_i)$$

$$\Rightarrow \quad Hx = \sum \alpha_i \langle x, e_i \rangle e_i$$

$$\Rightarrow \quad Tx = UHx = \sum \alpha_i \langle x, e_i \rangle Ue_i$$

f_i orthonormal since (e_j) orthonormal and U partial isometry with initial domain $\overline{\mathcal{R}(H)} \supset (e_j)$

$$\Rightarrow \quad T = \sum \alpha_i (e_i \otimes f_i)$$

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7.3.9 COROLLARY: Let $H \in B(\mathfrak{H})$ be Hermitian. Then H is compact if and only if

$$H = \sum \alpha_i \, e_i \otimes e_i$$

where (e_i) is an orthonormal sequence, $\alpha_i \in \mathbb{R}$, $\alpha_i \to 0$, and where the α_i's are the eigenvalues of H

7.3.10 THEOREM: $T \in B(\mathfrak{H}_1, \mathfrak{H}_2)$ is nuclear if and only if

$$T = \sum \lambda_i \, e_i \otimes f_i$$

where (e_i), (f_i) are orthonormal and $\lambda_i \geq 0$, $\sum \lambda_i < \infty$

Proof. ($\Leftarrow$) By definition

($\Rightarrow$) If T is nuclear, then $T = \sum \alpha_n \, x_n \otimes y_n$ where (x_n), (y_n) are unit vectors in $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively, and where $\alpha_i \geq 0$, $\sum \alpha_n < \infty$. Since T is compact,

$$T = \sum \lambda_i \, e_i \otimes f_i$$

(by 7.3.8) where (e_i), (f_i) are orthonormal, $\lambda_i \geq 0$, and $\lambda_i \to 0$
Want to show $\sum \lambda_i < \infty$. To this end, note that

$$\lambda_k = \langle Te_k, f_k \rangle = \langle \sum_n \alpha_n \langle e_k, x_n \rangle y_n, f_k \rangle$$

$$= \sum_n \alpha_n \langle e_k, x_n \rangle \langle y_n, f_k \rangle$$

$$\Rightarrow \lambda_k \leq \sum_n \alpha_n |\langle e_k, x_n \rangle \langle y_n, f_k \rangle|$$

$$\Rightarrow \sum_k \lambda_k \leq \sum_k \sum_n \alpha_n |\langle e_k, x_n \rangle| |\langle y_n, f_k \rangle|$$

$$= \sum_n \alpha_n \sum_k |\langle e_k, x_n \rangle| |\langle y_n, f_k \rangle|$$

$$\leq \sum_n \alpha_n \left(\sum_k |\langle e_k, x_n \rangle|^2 \right)^{1/2} \left(\sum_k |\langle y_n, f_k \rangle|^2 \right)^{1/2}$$

$$\leq \sum_n \alpha_n \|x_n\| \|y_n\| \quad \text{(Bessel's inequality)}$$

$$= \sum_n \alpha_n < \infty$$

$\boxtimes$

7.3.11 <u>DEFINITION</u>: An operator $T \in B(\mathfrak{H}_1, \mathfrak{H}_2)$ is Hilbert-Schmidt (HS) if there is an orthonormal basis (e_α) of $\mathfrak{H}_1$ such that

$$\sum_\alpha \|Te_\alpha\|^2 < \infty$$

Define Hilbert-Schmidt norm as $|||T||| := \left(\sum_\alpha \|Te_\alpha\|^2 \right)^{1/2}$

Denote the class by $HS(\mathfrak{H}_1, \mathfrak{H}_2)$.

7.3.12 **LEMMA**: The HS norm is independent of the choice of the orthonormal basis. Also, if T is HS, then T^* is HS and $|||T||| = |||T^*|||$
If U is a unitary operator, then

$$|||UT||| = |||T|||$$

$$|||TU||| = |||T|||$$

Moreover, $\|T\| \leq |||T|||$.

Proof. Let $\{e_\alpha : \alpha \in A\}$ and $\{f_\beta : \beta \in B\}$ be orthonormal bases of $\mathfrak{H}_1, \mathfrak{H}_2$

$$\|x\|^2 = \sum_\beta |\langle x, f_\beta \rangle|^2$$

$$\|Te_\alpha\|^2 = \sum_\beta |\langle Te_\alpha, f_\beta \rangle|^2$$

Hence

$$\sum \|Te_\alpha\|^2 = \sum_\alpha \sum_\beta |\langle Te_\alpha, f_\beta \rangle|^2$$

$$= \sum_\beta \sum_\alpha |\langle e_\alpha, T^* f_\beta \rangle|^2$$

$$= \sum_\beta \sum_\alpha |\langle T^* f_\beta, e_\alpha \rangle|^2$$

$$= \sum_\beta \|T^* f_\beta\|^2$$

Hence

$$(*) \qquad |||T|||_A = |||T^*|||_B$$

If A_1, A_2 are any basis in $\mathfrak{H}_1$, and B_1, B_2 are any basis in $\mathfrak{H}_2$, then $(*)$ shows that

$$|||T|||_{A_1} = |||T|||_{A_2} = |||T^*|||_{B_1} = |||T^*|||_{B_2}$$

In particular then, $|||TU||| = |||T|||$ since U takes basis to basis. Then

$$|||UT||| = |||(UT)^*||| = |||T^*U^*||| = |||T^*||| = |||T|||$$

Now $\forall \varepsilon > 0 \ \exists e_1, \ \|e_1\| = 1$, s.t. $\|T\|^2 \leq \|Te_1\|^2 + \varepsilon$
Imbed e_1 in an orthonormal basis. Then

$$\|T\|^2 \leq |||T|||^2 + \varepsilon$$

and so $\|T\|^2 \leq |||T|||^2$

$\blacksquare$

7.3.13 COROLLARY: If $T \in HS(\mathfrak{H}_1, \mathfrak{H}_2)$ and $(e_\alpha : \alpha \in A)$, $(f_\beta : \beta \in B)$ are orthonormal bases in $\mathfrak{H}_1$, $\mathfrak{H}_2$ respectively, then

$$|||T||| = \Big(\sum_{A,B} |\langle Te_\alpha, f_\beta \rangle|^2 \Big)^{1/2}$$

Proof $\quad \|Te_\alpha\|^2 = \sum'_{\beta \in B} |\langle Te_\alpha, f_\beta \rangle|^2$, and so

$$|||T|||^2 = \sum_\alpha \|Te_\alpha\|^2 = \sum_\alpha \sum'_\beta |\langle Te_\alpha, f_\beta \rangle|^2$$

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7.3.14 THEOREM

(a) $HS(\mathfrak{h}_1, \mathfrak{h}_2)$ is a B-space under the HS norm

(b) $HS(\mathfrak{h})$ is a B-algebra $(|||ST||| \le |||S|||\,|||T|||$

(c) $HS(\mathfrak{h})$ is a 2-sided ideal in $B(\mathfrak{h})$. If $T \in HS(\mathfrak{h})$ and $B \in B(\mathfrak{h})$, then

$$|||BT|| \le \|B\|\,|||T||| \,,\quad |||TB||| \le |||T|||\,\|B\|$$

Proof. (a) We have

$$|||T+S||| = \Big(\sum_{\alpha,\beta} |\langle (T+S)e_\alpha, f_\beta \rangle|^2 \Big)^{1/2}$$

$$= \Big(\sum_{\alpha,\beta} \big(|\langle Te_\alpha, f_\beta\rangle| + |\langle Se_\alpha, f_\beta \rangle| \big)^2 \Big)^{1/2}$$

$$\le \Big(\sum_{\alpha,\beta} |\langle Te_\alpha, f_\beta\rangle|^2 \Big)^{1/2} + \Big(\sum_{\alpha,\beta} |\langle Se_\alpha, f_\beta\rangle|^2 \Big)^{1/2}$$

$$= |||T||| + |||S|||$$

Norm since $\|T\| \le |||T|||$. Also, if $|||T_m - T_n||| \to 0$, then $\|T_m - T_n\| \to 0$

so $T_n \to T$ in $\mathcal{B}(\mathcal{H})$. It is easily seen that T is Hilbert-Schmidt and $|||T_n - T||| \to 0$ (Argument similarly to completeness of ℓ_2)

(c) $\sum \| BTe_\alpha \|^2 \le \sum \|B\|^2 \|Te_\alpha\|^2 = \|B\|^2 \sum \|Te_\alpha\|^2$

$$\Rightarrow \quad |||BT|||^2 \le \|B\|^2 |||T|||^2$$

Hence BT is HS. Also, $(TB)^* = B^* T^*$ is HS since T^* is HS and by first result, and so TB is HS, with

$$|||TB||| = |||(TB)^*||| = |||B^* T^*||| \le \|B^*\| \, |||T^*|||$$

$$= \|B\| \, |||T|||$$

(b) follows directly from (c).

$\boxtimes$

7.3.15 <u>THEOREM</u>: A Hilbert-Schmidt operator is compact. In fact, it is the limit in the HS norm of a sequence of finite rank operators.

Proof. We know $|||T|||^2 = \sum_\alpha \|Te_\alpha\|^2 < \infty$. Given $\varepsilon > 0$,

$\exists\, A_m$ (finite) $\subset A$ s.t. $\sum_{\alpha \notin A_m} \|Te_\alpha\|^2 < \varepsilon^2$. Let

$$T_n e_\alpha = \begin{cases} Te_\alpha & \alpha \in A_n \\ 0 & \alpha \notin A_n \end{cases}$$

and extend by linearity and continuity. T_n has finite rank and $T_n \in HS$

$$|||T - T_n|||^2 = \sum_{\alpha \in A_m} \|Te_\alpha\|^2 < \varepsilon^2$$

$\blacksquare$

7.3.16 <u>THEOREM</u>: A compact operator $T \in B(\mathfrak{H}_1, \mathfrak{H}_2)$ is HS if and only if it has a representation

$$(*) \qquad T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes f_i$$

where $(e_i) \subset \mathfrak{H}_1$, $(f_i) \subset \mathfrak{H}_2$ are orthonormal sequences and where $\lambda_i \geq 0$ satisfy $\sum \lambda_i^2 < \infty$.

Proof. If T is HS, then it necessarily is compact and has a representation $(*)$ with $0 \leq \lambda_i \to 0$. Then

$$Te_k = \lambda_k f_k$$

$$\Rightarrow \|Te_k\|^2 = \lambda_k^2$$

$$\Rightarrow \sum' \lambda_k^2 = \sum \|Te_k\|^2 < \infty$$

Conversely, suppose T has the representation in the form $(*)$ with $\sum \lambda_i^2 < \infty$. Expand (e_i) to an orthonormal basis $(\hat{e}_\alpha)$. If $\hat{e}_\alpha$ is a "new" element, $\tilde{e}_\alpha \perp e_i \ \forall i$

$$T\hat{e}_\alpha = \sum \lambda_i \langle \hat{e}_\alpha, e_i \rangle f_\alpha = 0$$

$$Te_j = \lambda_j f_j$$

Hence

$$\sum \|T\hat{e}_\alpha\|^2 = \sum_j \|Te_j\|^2 = \sum_j \lambda_j^2 < \infty$$

Hence T is HS (since $\hat{e}_\alpha$ basis)

$\blacksquare$

7.3.17 **THEOREM:** $T \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is HS iff T^*T is nuclear

Proof. ($\Rightarrow$) If $T \in HS$, then

$$T = \sum \lambda_i e_i \otimes f_i$$

where $\sum \lambda_i^2 < \infty$. Then

$$T^* = \sum \lambda_i f_i \otimes e_i$$

Similar calculations show that

$$T^*T = \sum \lambda_i^2 e_i \otimes e_i = \sum \mu_i e_i \otimes e_i$$

where $\sum \mu_i = \sum \lambda_i^2 < \infty$. Hence T^*T is nuclear in $B(\mathcal{H}_1)$

$(\Leftarrow)$ Suppose T^*T is nuclear. Certainly $T^*T \geq 0$ since

$$\langle T^*Tx, x\rangle_1 = \langle Tx, Tx\rangle_2 = \|Tx\|_2^2 \geq 0$$

Also T^*T is compact, and so by Cor 7.3.9

$$T^*T = \sum \mu_i \, e_i \otimes e_i \qquad \text{where } 0 \leq \mu_i \to 0$$

orthonormal

Since T^*T is nuclear, $\sum \mu_i < \infty$ (by argument of 7.3.10)

Claim: $\sqrt{T^*T} = \sum \sqrt{\mu_i} \, e_i \otimes e_i$

But $T = U\sqrt{T^*T}$, so

$$T = \sum \sqrt{\mu_i} \, e_i \otimes Ue_i$$

ε_i orthonormal since $\varepsilon_i \in \overline{R(\sqrt{T^*T})}$

unitary
partial isometry
initial domain $\overline{R(\sqrt{T^*T})}$

Since $\sum (\sqrt{\mu_i})^2 = \sum \mu_i < \infty$, T is HS by theorem 7.3.16

$\blacksquare$

3/18 SPECTRAL THEORY

COROLLARY: If $S, T \in HS(\mathfrak{H})$, then $ST \in \eta(\mathfrak{H})$

Proof. Since $S, T \in HS(\mathfrak{H})$, we have $S+T$ and $S+iT$ are HS. Hence $(S+T)^*$ and $S^* - iT^*$ are also HS. Therefore

$$(S+T)^*(S+T) \in \eta$$

$$\Rightarrow \quad S^*S + S^*T + T^*S + T^*T \in \eta$$

$$\Rightarrow \quad S^*T + T^*S \in \eta$$

since $S^*S, T^*T \in \eta$

and also

$$(S^* - iT^*)(S + iT) \in \eta$$

$$\Rightarrow \quad S^*S + iS^*T - iT^*S + T^*T \in \eta$$

$$\Rightarrow \quad S^*T - T^*S \in \eta$$

Hence $S^*T \in \eta$ for any $S, T \in HS$. But $S \in HS \Rightarrow S^* \in HS$, and so $ST \in \eta$ for any S, T

∎

References: 1) D.S. Part II XI.6, XI.9
2) I.C. Gohberg - M.S. Krein Introd. to non self-adjoint operators
AMS Translations of Math Monographs, vol. 18 (1969)
3) J.R. Ringrose Compact non self-adjoint operators
Van Nostrand Math Studies, vol 35 (1971)

7.3.19 <u>THEOREM</u>: (a) If $T \in \eta(\mathfrak{H})$ is Hermitian and has eigenvalues (λ_i) counted according to multiplicity (c.a.m), then $\sum |\lambda_i| < \infty$

(b) If (e_α) is an orthonormal basis for $\mathfrak{H}$ and T as above, then

$$\sum_\alpha \langle Te_\alpha, e_\alpha \rangle = \sum_{i=1}^{\infty} \lambda_i$$

where the convergence is absolute

Proof. a) Since $T^* = T$, Corollary 7.3.9 implies

$$T = \sum \lambda_i \, x_i \otimes x_i$$

where (x_i) is orthonormal and λ_i are c.a.m. Since T is nuclear,

$$T = \sum \alpha_i \, u_i \otimes v_i$$

where $\sum \alpha_i < \infty$, $\alpha_i \geq 0$, $\|u_i\| = \|v_i\| = 1$.

Now

$$\lambda_k = \langle Tx_k, x_k \rangle = \langle \sum \alpha_i \langle x_k, u_i \rangle v_i , x_k \rangle$$

$$= \sum \alpha_i \langle x_k, u_i \rangle \langle v_i, x_k \rangle$$

$$\Rightarrow |\lambda_k| \leq \sum |\alpha_i| \, |\langle x_k, u_i \rangle| \, |\langle v_i, x_k \rangle|$$

$$\Rightarrow \sum_{k=1}^{\infty} |\lambda_k| \leq \sum_{i=1}^{\infty} \alpha_i \sum_{k=1}^{\infty} |\langle x_k, u_i \rangle| \, |\langle v_i, x_k \rangle|$$

$$\leq \sum_{i=1}^{\infty} \alpha_i \left(\sum_{k=1}^{\infty} |\langle x_k, u_i \rangle|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |\langle v_i, x_k \rangle|^2 \right)^{1/2}$$

$$\leq \sum_{i=1}^{\infty} \alpha_i \, \|u_i\| \, \|v_i\|$$

$$= \sum_{i=1}^{\infty} \alpha_i < \infty$$

(b) Let (e_α) be an orthonormal basis. With T as in (a)

$$Te_\alpha = \sum \lambda_i \langle e_\alpha, x_i \rangle x_i$$

$$\therefore \langle Te_\alpha, e_\alpha \rangle = \sum \lambda_i |\langle e_\alpha, x_i \rangle|^2 \quad (*)$$

$$\therefore |\langle Te_\alpha, e_\alpha \rangle| \leq \sum |\lambda_i| \, |\langle e_\alpha, x_i \rangle|^2$$

We thus obtain

$$\sum_\alpha |\langle Te_\alpha, e_\alpha\rangle| \leq \sum_i |\lambda_i| \left(\sum_\alpha |\langle e_\alpha, x_i\rangle|^2 \right)$$

$$= \sum_i |\lambda_i| \, \|x_i\|^2$$

$$= \sum_i |\lambda_i| < \infty$$

Hence $\sum_\alpha \langle Te_\alpha, e_\alpha\rangle$ is absolutely convergent. Therefore

$$\sum_\alpha \langle Te_\alpha, e_\alpha\rangle = \sum_\alpha \sum_i \lambda_i |\langle e_\alpha, x_i\rangle|^2$$

since abs. conv.

$$= \sum_i \sum_\alpha \lambda_i |\langle e_\alpha, x_i\rangle|^2$$

$$= \sum_i \lambda_i \|x_i\|$$

$$= \sum_i \lambda_i$$

$\blacksquare$

7.3.20 THEOREM: If $T \in \eta(\mathcal{H})$ and (e_α) is any orthonormal basis for $\mathcal{H}$, then the series

$$\sum_\alpha \langle Te_\alpha, e_\alpha \rangle$$

converges absolutely. This sum does not depend on the choice of the basis.

Proof. $T = A + iB$ where

$$A = \frac{1}{2}(T+T^*)$$

$$B = \frac{1}{2}(T-T^*)$$

are Hermetian nuclear operators. Let $(\alpha_i), (\beta_i)$ be eigenvalues of A, B respectively. Then

$$\sum \langle Te_\alpha, e_\alpha \rangle = \sum \langle Ae_\alpha, e_\alpha \rangle + i \sum \langle Be_\alpha, e_\alpha \rangle$$

$$= \sum \alpha_j + i \sum \beta_j$$

and this sum is abs. convergent.

$\boxtimes$

Remark: In fact, if $T \in \eta(\mathcal{H})$, then $\sum \langle Te_\alpha, e_\alpha \rangle = \sum \lambda_i$ where (λ_i) are the eigenvalues of T c.a.m.. [[Ringrose Th 3.3.13 p139]]

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7.3.21 DEFINITION: If T is a nuclear operator on $\mathfrak{H}$, define

$$\operatorname{tr} T = \sum \langle Te_\alpha, e_\alpha \rangle \quad \left(= \sum \lambda_n\right)$$

(trace of T) where (e_α) is any orthonormal basis.

7.3.22 THEOREM: If $T, S \in \eta(\mathfrak{H})$, $a, b \in \mathbb{C}$, $B \in \mathcal{B}(\mathfrak{H})$, then

(a) $\operatorname{tr}(aT + bS) = a \operatorname{tr} T + b \operatorname{tr} S$

(b) $\operatorname{tr}(S^*) = \operatorname{tr}(S)$

(c) If $S \geq 0$, then $\operatorname{tr} S \geq 0$. If $S \geq 0$ and $\operatorname{tr} S = 0$, then $S = 0$

(d) $\operatorname{tr}(BS) = \operatorname{tr}(SB)$

Proof. (a), (b), (c) clear from definition

(d) Let $B = U$ be unitary. Then

$$\operatorname{tr}(SU) = \sum \langle SUe_\alpha, e_\alpha \rangle = \sum \langle SUe_\alpha, U^*Ue_\alpha \rangle$$

$$= \sum \langle USUe_\alpha, Ue_\alpha \rangle = \operatorname{tr}(US)$$

since (Ue_α) is another orthonormal basis.

To prove the general case when B is not unitary we use the following lemma.

7.2.23 <u>LEMMA</u>: Every $B \in B(\mathcal{H})$ is a finite linear combination of unitary operators.

Proof. Write $B = A_1 + i A_2$ where $A_i^* = A_i$. Treat case $B = B^*$, $\|B\| \le 1$. Then $\sigma(B) \subset [-1,1]$. Let

$$f(t) := t + i\sqrt{1-t^2}$$

for $f \in C[-1,1]$. Then

$$(*) \qquad f(t) f(t)^* = t^2 + (1-t^2) = 1$$

Note that

$$t = \frac{1}{2}\left(f(t) + f(t)^*\right)$$

Let $U := f(B)$. By $(*)$, $UU^* = I = U^*U$. We also have

$$B = \frac{1}{2}(U + U^*) \qquad \blacksquare$$

Returning to the proof of (d), write B as a finite linear combination of unitary operators and use the previous result. $\blacksquare$

7.3.24 <u>DEFINITION</u>: If $S, T \in HS(\mathfrak{H})$, define

$$[S,T] := \operatorname{tr}(T^*S)$$

(Well-defined since T^*S is nuclear)

7.3.25 <u>THEOREM</u>: The function $[\cdot,\cdot]$ is an inner product on $HS(\mathfrak{H})$ such that

$$|||T|||^2 = [T,T]$$

Thus $HS(\mathfrak{H})$ is a Hilbert space under this inner product (In particular,

$$|[S,T]| \leq |||S|||\ |||T|||$$)

Moreover, the map $S \mapsto S^*$ satisfies

$$(*) \qquad [ST,R] = [T, S^*R]$$

Proof. $[T,T] = \operatorname{tr}(T^*T) = \sum \langle T^*Te_\alpha, e_\alpha\rangle = \sum \|Te_\alpha\|^2 = |||T|||^2$
Rest clear.

W. Ambrose [TAMS 57 (1945)] H^*-alg. Hilbert space B-algebra satisfying (*)

7.3.25 THEOREM: If $T \in HS(\mathcal{H})$,

$$|||T||| = \left(\sum \lambda_n^2\right)^{1/2}$$

where (λ_n) are the eigenvalues of $\sqrt{T^*T} = |T|$ counted according to multiplicity.

Proof. $0 \le T^*T \in \eta$. Then

$$tr(T^*T) = \sum \lambda_n^2$$

But $tr(T^*T) = [T,T] = |||T|||^2$

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§7.4 FREDHOLM OPERATORS AND INDEX

7.4.1 DEFINITION: Let $\mathcal{X}, \mathcal{Y}$ be B-spaces, $T \in B(\mathcal{X}, \mathcal{Y})$. Suppose that $\mathrm{Rng}(T)$ is closed in $\mathcal{Y}$.

(a) Let $\alpha(T) := \dim \eta(T)$ if $\dim \eta(T) < \infty$ ("nullity")
$= +\infty$ otherwise

(b) Let $\beta(T) := \dim \mathcal{Y}/R(T)$ if finite ("defect" or "deficiency")
$= +\infty$ otherwise

$= \operatorname{codim} R(T)$

(c) We say that T is a Fredholm operator if $R(T)$ is closed and both $\alpha(T)$ and $\beta(T)$ are finite. Define the index of T to be

$$\operatorname{ind}(T) := \alpha(T) - \beta(T) \in \mathbb{Z}$$

Write $\Phi(\mathcal{X}, \mathcal{Y})$ for the set of all Fredholm operators

(d) $T \in \Phi_+(\mathcal{X}, \mathcal{Y})$ if $R(T)$ is closed and $\alpha(T) < \infty$
$T \in \Phi_-(\mathcal{X}, \mathcal{Y})$ if $R(T)$ is closed and $\beta(T) < \infty$

In either case, we say that T is semi-Fredholm and define the index as in (c). Then $\operatorname{ind}(T) \in \mathbb{Z} \cup \{\pm\infty\}$.

Examples of Fredholm operators:

a) invertible operators
b) $\lambda I - K$ for $\lambda \neq 0$, K compact (Here $\operatorname{ind}(\lambda I - K) = 0$

7.4.2 LEMMA: If $\mathcal{R}(T)$ is closed, then

$$\alpha(T^*) = \beta(T) \qquad \beta(T^*) = \alpha(T)$$

$$\alpha(T) = \alpha(T^{**}) \qquad \beta(T) = \beta(T^{**})$$

Proof. By theorem 5.2.4, $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$. Since $\mathcal{N}(T)$ is closed, $(\mathcal{N}(T))^* = \mathfrak{X}^*/_{\mathcal{N}(T)^\perp}$, and so

$$\mathcal{N}(T)^* = \mathfrak{X}^*/_{\mathcal{R}(T^*)}$$

Then

$$\alpha(T) = \dim \mathcal{N}(T) = \dim \mathcal{N}(T)^* = \dim \left(\mathfrak{X}^*/_{\mathcal{R}(T^*)}\right)$$

since finite or both so

$$= \beta(T^*)$$

Similarly,

$$\left(\mathcal{Y}/\mathcal{R}(T) \right)^* = \mathcal{R}(T)^\perp \underset{\uparrow \, 5.2.4(a)}{=} \eta(T^*)$$

and so

$$\beta(T) = \dim\left(\mathcal{Y}/\mathcal{R}(T)\right) = \dim\left(\mathcal{Y}/\mathcal{R}(T)\right)^* = \dim \eta(T^*)$$

$$= \alpha(T^*)$$

Finally, $\mathcal{R}(T)$ closed $\Rightarrow \mathcal{R}(T^*)$ closed $\Rightarrow \mathcal{R}(T^{**})$ closed

and

$$\alpha(T) = \beta(T^*) = \alpha(T^{**})$$

$$\beta(T) = \alpha(T^*) = \beta(T^{**})$$

$\boxtimes$

7.4.3 **Corollary:** $T \in \Phi_+ \iff T^* \in \Phi_-$

$T \in \Phi_- \iff T^* \in \Phi_+$

$T \in \Phi \iff T^* \in \Phi$

$\operatorname{ind}(T) = -\operatorname{ind}(T^*)$

7.4.4. **LEMMA:** If $\mathcal{U}_0$ is a finite dimensional subspace of $\mathcal{X}$, then there exists a subspace $\mathcal{U}_1$ of $\mathcal{X}$ with

$$\mathcal{X} = \mathcal{U}_0 \oplus \mathcal{U}_1$$

Proof. Let $\{u_1, \dots, u_n\}$ be a basis in $\mathcal{U}_0$. Let $\{x_1^*, \dots, x_n^*\}$ be a dual basis (of $\mathcal{U}_0^*$ but extended to $\mathcal{X}^*$)

$$x_i^* \, u_j = \delta_{ij}$$

Let $\mathcal{U}_1 := \{x_1^*, \dots, x_n^*\}_\perp \subset \mathcal{X}$. Then $\mathcal{U}_1$ is a subspace of $\mathcal{X}$.

Claim: $\mathcal{U}_1 \cap \mathcal{U}_0 = \{0\}$. Suppose $u = \sum \alpha_i u_i \in \mathcal{U}_0$. If $u \in \mathcal{U}_1$, then $x_j^* u = 0 \ \forall j$, i.e.

$$\alpha_j = x_j^* \left(\sum \alpha_i u_i \right) = 0$$

and so $u = 0$.

Let $P := \sum x_k^* \otimes u_k$. Then $Px = \sum (x_k^* x) u_k \in \mathcal{U}_0$. If $u \in \mathcal{U}_0$, $u = \sum \alpha_k u_k$, then

$$Pu = \sum \alpha_k P u_k = \sum \alpha_k u_k = u$$

and

$$x_j^* (x - Px) = x_j^* x - \sum (x_k^* x) \, x_j^* (u_k) = x_j^* x - x_j^* (x) = 0$$

Hence $x - Px \in \mathcal{U}_1$, so

$$x = Px + (x - Px) \in \mathcal{U}_0 + \mathcal{U}_1$$

$\boxtimes$

7.4.5. LEMMA: Let $\mathfrak{X}$ be a Banach space, $\mathcal{R}$ a subspace of $\mathfrak{X}$. If $\mathfrak{X}/\mathcal{R}$ is n-dimensional, then there exists an n-dimensional subspace $\mathcal{V} \subset \mathfrak{X}$ with $\mathfrak{X} = R \oplus \mathcal{V}$

Proof Let $\{z_1, \ldots, z_n\}$ be such that $\{[z_1], \ldots, [z_n]\}$ is a basis for $\mathfrak{X}/R$. Then $\{z_1, \ldots, z_n\}$ is linearly independent. Let $\mathcal{V} = \operatorname{span}\{z_1, \ldots, z_n\}$

Claim: $\mathcal{V} \cap \mathcal{R} = \{0\}$. If $x = \sum \alpha_i z_i \in \mathcal{V} \cap \mathcal{R}$, then

$$0 = \sum \alpha_i [z_i] \quad (\text{since in } \mathcal{R})$$

$$\Rightarrow \alpha_i = 0 \ \forall i \Rightarrow x = 0$$

Claim: $\mathfrak{X} = \mathcal{V} + \mathcal{R}$. Let $x \in \mathfrak{X}$. Look at $[x] \in \mathfrak{X}/R$

Then

$$[x] = \sum \alpha_i [z_i]$$

If $v := \sum \alpha_i z_i \in \mathcal{V}$, then $r = v - x \in R$ since $[v] = [x]$, so $x = v + r$. $\boxtimes$

7.4.6 <u>COROLLARY</u>: There exists a continuous projection $Q : \mathcal{X} \to \mathcal{R}$ with range $\mathcal{R}$ and null space $\mathcal{V}$ [Hypotheses as in last lemma]

3/25 SPECTRAL THEORY

References: J. Dieudonné (Bull. Sci. Math Fr. 67 (1943))
B. Yood Duke J. 18 (1951)
F.V. Atkinson Math Sbornik
I.C. Gohberg - M.S. Krein, AMS Translation (2) 13 (1960) p185
T. Kato (J. d'Analyse Math 6 (1958) p261

7.4.7 <u>THEOREM (Atkinson)</u>: If $T \in \Phi(\mathfrak{X}, \mathcal{Y})$, then there exists a closed subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ and a $\beta(T)$-dimensional subspace $\mathcal{Y}_0$ of $\mathcal{Y}$ s.t.

$$\mathfrak{X} = \mathfrak{X}_0 \oplus \mathfrak{N}(T) \qquad \mathcal{Y} = \mathfrak{R}(T) \oplus \mathcal{Y}_0$$

Moreover, there exists $S \in \mathcal{B}(\mathcal{Y}, \mathfrak{X})$ s.t.

$$\mathfrak{N}(S) = \mathcal{Y}_0 \qquad \mathfrak{R}(S) = \mathfrak{X}_0$$

$$ST = I_{\mathfrak{X}_0} \qquad TS = I_{\mathfrak{R}(T)}$$

Proof. Since $\mathfrak{N}(T)$ is finite dimensional, by lemma 7.4.4 $\exists\, \mathfrak{X}_0$ such that $\mathfrak{X} = \mathfrak{X}_0 \oplus \mathfrak{N}(T)$. Similarly, $\mathcal{Y}/\mathfrak{R}(T)$ is finite dimensional so by lemma 7.4.5 $\exists\, \mathcal{Y}_0$ with $\mathcal{Y} = \mathcal{Y}_0 \oplus \mathfrak{R}(T)$.

Claim: $T|\mathcal{X}_0$ is one-to-one. This follows since $\mathcal{N}(T) \cap \mathcal{X}_0 = \{0\}$.

Let $T_1 := T|\mathcal{X}_0$. Then $T_1 \in B(\mathcal{X}_0, \mathcal{R}(T))$. In fact $\mathcal{R}(T_1) = \mathcal{R}(T)$. For if $y \in \mathcal{R}(T)$, then $y = Tx$ for some x. Write $x = x_0 + n$, where $x_0 \in \mathcal{X}_0$ and $n \in \mathcal{N}(T)$. Then $y = Tx = Tx_0 = T_1 x_0$. Therefore

$$S_1 := T_1^{-1} \in B(\mathcal{R}(T), \mathcal{X}_0)$$

By corollary 7.4.6, there exists a continuous projection $P: \mathcal{Y} \to \mathcal{R}(T)$ with $\mathcal{N}(P) = \mathcal{Y}_0$. Let

$$S := S_1 P = T_1^{-1} P$$

Then $S \in B(\mathcal{Y}, \mathcal{X})$.
Rest clear.

$\blacksquare$

7.4.8. COROLLARY: With assumptions as above,

$$ST = I_{\mathcal{X}} - F_1$$

$$TS = I_{\mathcal{Y}} - F_2$$

where F_1 and F_2 are finite rank operators in $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively.

Proof. $(ST - I)x = 0$ for $x \in \mathcal{X}_0$. Since $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{N}(T)$ and $\mathcal{N}(T)$ has finite dimension, $ST - I$ has finite dimensional range

Similar argument for $TS - I_{\mathcal{Y}}$.

$\blacksquare$

7.4.9. <u>LEMMA:</u> Let $\mathcal{X}$ be a normed linear space with $\mathcal{X} = \mathcal{N} \oplus \mathcal{X}_0$ where $\mathcal{N}$ is finite dimensional and $\mathcal{X}_0$ is closed. If $\mathcal{X}_1$ is a linear manifold in $\mathcal{X}$ and $\mathcal{X}_0 \subset \mathcal{X}_1$, then $\mathcal{X}_1$ is closed.

Proof. Let $\mathcal{M} := \mathcal{N} \cap \mathcal{X}_1$. This is a finite dimensional subspace and so it is closed.

Claim: $\mathcal{X}_1 = \mathcal{M} \oplus \mathcal{X}_0$. Certainly $\mathcal{M} \cap \mathcal{X}_0 = \mathcal{N} \cap \mathcal{X}_0 \cap \mathcal{X}_1 = \{0\}$. Now suppose $x \in \mathcal{X}_1$. Then $x = n + x_0$ where $n \in \mathcal{N}$ and $x_0 \in \mathcal{X}_0$. Then $n = x - x_0 \in \mathcal{X}_1$, and $n \in \mathcal{N}$, so $n \in \mathcal{M}$. Since $x_0 \in \mathcal{X}_1$ also, we see that $\mathcal{X}_1 = \mathcal{M} + \mathcal{X}_0$.

Since $\mathcal{M}$ and $\mathcal{X}_0$ are closed, it follows that $\mathcal{X}_1$ is closed.
(and $\mathcal{M}$ is f.d.)

$\blacksquare$

7.4.10 <u>THEOREM:</u> Let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and suppose there exists $S_1, S_2 \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and compact operators $K_1 \in \mathcal{K}(\mathcal{X})$ and $K_2 \in \mathcal{K}(\mathcal{Y})$ s.t.

$$S_1 T = I_{\mathcal{X}} - K_1 \qquad TS_2 = I_{\mathcal{Y}} - K_2$$

Then $T \in \Phi(\mathcal{X}, \mathcal{Y})$.

Proof. $\eta(T) \subset \eta(S_1 T) = \eta(I_{\mathcal{X}} - K_1)$; $\dim \eta(I_{\mathcal{X}} - K_1) < \infty$
since K_1 is compact ($1 \in \sigma(K_1) \Rightarrow$ eigenvalue of finite multiplicity)

$R(T) \supset R(TS_2) = R(I - K_2)$; closed, has finite codimension
Hence $R(T)$ is closed (by lemma), and has finite codimension

$$\beta(T) = \dim \mathcal{Y}/_{R(T)} \leq \dim \mathcal{Y}/_{R(I-K_2)} < \infty$$

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7.4.11 <u>Index Theorem</u>: Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be B-spaces. Let $T \in \Phi(\mathcal{X}, \mathcal{Y})$, $S \in \Phi(\mathcal{Y}, \mathcal{Z})$. Then $ST \in \Phi(\mathcal{X}, \mathcal{Z})$ and

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$$

Proof. $\exists\, T_1 \in \Phi(\mathcal{Y}, \mathcal{X})$, $S_1 \in \Phi(\mathcal{Z}, \mathcal{Y})$ s.t.

$$T_1 T = I - F_1 \qquad T T_1 = I - F_2$$
$$S_1 S = I - F_3 \qquad S S_1 = I - F_4$$

Then

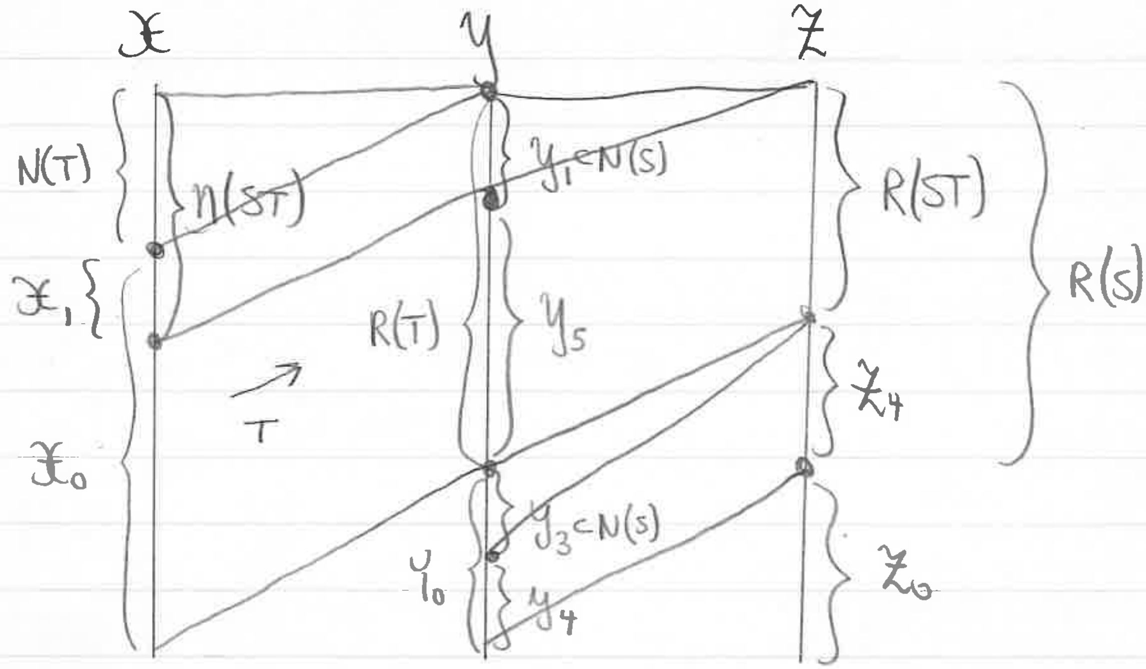
$$(T_1 S_1)(ST) = T_1 (I - F_3) T = T_1 T - T_1 F_3 T$$

$$= I - \underbrace{F_1 - T_1 F_3 T}_{\text{finite rank}} = I - F_5$$

$$(ST)(T_1 S_1) = S(I - F_2) S_1 = S S_1 - S F_2 S_1$$

$$= I - \underbrace{F_3 - S F_2 S_1}_{\text{finite rank}} = I - F_6$$

Hence by Theorem 7.4.10 $ST \in \Phi(\mathcal{X}, \mathcal{Z})$



Let $\mathcal{Y}_1 := \mathcal{R}(T) \cap \mathcal{N}(S)$ f.d.

$\exists\, \mathcal{Y}_2 \subset \mathcal{R}(T),\ \mathcal{Y}_3 \subset \mathcal{N}(S)$ s.t. (i) $\mathcal{R}(T) = \mathcal{Y}_1 \oplus \mathcal{Y}_2$

(ii) $\mathcal{N}(S) = \mathcal{Y}_1 \oplus \mathcal{Y}_3$

Since $\mathcal{Y}_3$ is f.d. $\mathcal{R}(T) \oplus \mathcal{Y}_3$ is closed and has finite codimension. Therefore $\exists\, \mathcal{Y}_4$ f.d. such that

(iii) $$\mathcal{Y} = \mathcal{R}(T) \oplus \mathcal{Y}_3 \oplus \mathcal{Y}_4$$

Let $\mathfrak{X}_1 := \mathcal{N}(ST) \cap \mathfrak{X}_0$ f.d. Then

(iv) $$\mathcal{N}(ST) = \mathcal{N}(T) \oplus \mathfrak{X}_1$$

Let $\mathcal{Z}_4 := S(\mathcal{Y}_4)$. Then

(v) $\qquad \mathcal{R}(S) = \mathcal{R}(ST) \oplus \mathcal{Z}_4$

For suppose $z \in \mathcal{R}(ST) \cap \mathcal{Z}_4 = S\mathcal{R}(T) \cap \mathcal{Z}_4 = S(\mathcal{Y}_1 \oplus \mathcal{Y}_2) \cap \mathcal{Z}_4$

$$= S\mathcal{Y}_2 \cap \mathcal{Z}_4 = S\mathcal{Y}_2 \cap S\mathcal{Y}_4$$

Then $z = Sy_2 = Sy_4 \Rightarrow y_2 - y_4 \in \mathcal{N}(S) = \mathcal{Y}_1 \oplus \mathcal{Y}_3$
But $y_2 - y_4 \in \mathcal{Y}_2 \oplus \mathcal{Y}_4$ and $\mathcal{Y}_1 \oplus \mathcal{Y}_3 \cap \mathcal{Y}_2 \oplus \mathcal{Y}_4 = \{0\}$ Hence $y_2 = y_4$
But $\mathcal{Y}_2 \cap \mathcal{Y}_4 = \{0\}$, and so $y_2 = y_4 = 0 \Rightarrow z = 0$. Now let $z \in \mathcal{R}(S)$. Then $z = Sy$, $y \in \mathcal{Y}$. Since

$$y = y_1 + y_2 + y_3 + y_4$$

we have $z = Sy = Sy_2 + Sy_4 \in \mathcal{R}(ST) + \mathcal{Z}_4$

CLAIM: T is one-to-one of $\mathfrak{X}_1$ onto $\mathcal{Y}_1$
S is " " $\mathcal{Y}_4$ onto $\mathcal{Z}_4$

Proof. Let $y_1 \in \mathcal{Y}_1 = \mathcal{R}(T) \cap \mathcal{N}(S)$. $\exists\, \bar{x}$ with $y_1 = T\bar{x}$
But

$$\bar{x} = n + x_0 \qquad n \in \mathcal{N}(T),\ x_0 \in \mathfrak{X}_0$$

$$\Rightarrow y_1 = T\bar{x} = Tx_0$$

But $0 = Sy_1 = STx_0 \Rightarrow x_0 \in \mathcal{N}(ST) \cap \mathfrak{X}_0 = \mathfrak{X}_1$. Hence
$T : \mathfrak{X}_1 \to \mathcal{Y}_1$ is one-to-one onto (1-1 since $\mathfrak{X}_1 \cap N(T) = (0)$)

$y_4 \in \mathcal{Y}_4$

If $Sy_4 = 0$, then $y_4 \in \mathcal{Y}_1 \oplus \mathcal{Y}_3 \Rightarrow y_4 = 0$. S is clearly onto $\mathcal{Z}_4$ from $\mathcal{Y}_4$.

Hence $\dim \mathcal{X}_1 = \dim \mathcal{Y}_1$ and $\dim \mathcal{Y}_4 = \dim \mathcal{Z}_4$

$$
\begin{aligned}
&(iv) \Rightarrow \alpha(ST) = \alpha(T) + \dim \mathcal{X}_1 \\
&(v) \Rightarrow \beta(ST) = \beta(S) + \dim \mathcal{Z}_4 \\
&(iii) \Rightarrow \beta(T) = \dim \mathcal{Y}_3 + \dim \mathcal{Y}_4 \\
&(ii) \Rightarrow \alpha(S) = \dim \mathcal{Y}_1 + \dim \mathcal{Y}_3
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{ind}(ST) = \alpha(ST) - \beta(ST) &= \alpha(T) + \dim \mathcal{X}_1 - \beta(S) - \dim \mathcal{Z}_4 \\
&= \alpha(T) + \dim \mathcal{Y}_1 - \beta(S) - \dim \mathcal{Y}_4 \\
&= \alpha(T) + \alpha(S) - \dim \mathcal{Y}_3 - \beta(S) - \beta(T) + \dim \mathcal{Y}_3 \\
&= \alpha(T) - \beta(T) + \alpha(S) - \beta(S) \\
&= \operatorname{ind}(T) + \operatorname{ind}(S)
\end{aligned}
$$

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7.4.12 THEOREM: Let $T \in \Phi(\mathcal{X},\mathcal{Y})$, $K \in \mathcal{K}(\mathcal{X},\mathcal{Y})$. Then $T+K \in \Phi$ and $\operatorname{ind}(T+K) = \operatorname{ind} T$

Proof. By 7.4.7 $\exists\, T_1 \in \Phi(\mathcal{Y},\mathcal{X})$ s.t.

$$T_1 T = I - F_1 \qquad T T_1 = I - F_2$$

Then

$$T_1(T+K) = I - \underbrace{F_1 + T_1 K}_{\text{compact}} \in \Phi(\mathcal{X},\mathcal{X}) \Rightarrow T+K \in \Phi$$

and

$$\operatorname{ind} T_1 + \operatorname{ind}(T+K) = \operatorname{ind}(T_1(T+K)) = 0$$

But

$$\operatorname{ind} T_1 + \operatorname{ind} T = \operatorname{ind}(T_1 T) = \operatorname{ind}(I - F_1) = 0$$

and so $\operatorname{ind}(T+K) = \operatorname{ind} T$

▨

7.4.13 Example: (a) Let $K \in B(\ell_\infty)$ be given by

$$K(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, \ldots)$$

$$(I-K)(x_1,x_2,\dots) = (0,x_2,x_3,\dots)$$

$$\Rightarrow \alpha(I-K)=1 \qquad \beta(I-K)=1$$

Since $\alpha(I)=0=\beta(I)$, you have increased both α and β by perturbing by K.

7.4.14 <u>THEOREM:</u> Let $T\in\Phi(\mathcal{X},\mathcal{Y})$. There exists $\gamma>0$ such that if $\|B\|<\gamma$, $B\in B(\mathcal{X},\mathcal{Y})$, then $T+B\in\Phi(\mathcal{X},\mathcal{Y})$ and $\operatorname{ind}(T+B)=\operatorname{ind}T$. Moreover

$$\alpha(T+B)\le\alpha(T)$$
$$\beta(T+B)\le\beta(T)$$

Proof. $\exists\ S\in\Phi(\mathcal{Y},\mathcal{X})$ s.t.

$$ST=I-F_1 \qquad TS=I-F_2$$

Then $S(T+B) = I-F_1+SB$ and $(T+B)S = I-F_2+BS$
$= I+SB-F_1$ $\qquad\qquad = I+BS-F_2$

If $\|B\|<1/\|S\|$, then $I+BS$ and $I+SB$ are invertible. (Let $\gamma=\|S\|^{-1}$.) Then

$$((I+SB)^{-1}S)(T+B) = I-\underbrace{(I+SB)^{-1}F}_{\text{finite rank}}\in\Phi$$

and

$$(T+B)\left(S(I+BS)^{-1}\right) = I - \underbrace{F_2(I+BS)^{-1}}_{\text{finite rank}} \in \Phi$$

Hence $T+B$ is Fredholm by 7.4.10

Since $I+SB$ and $I+BS$ are invertible, they are Fredholm operators. Then

$$\cancelto{0}{\operatorname{ind}\left((I+SB)^{-1}\right)} + \operatorname{ind} S + \operatorname{ind}(T+B) = \operatorname{ind}\left(I-(I+SB)^{-1}F\right) = 0$$

$$\Rightarrow \operatorname{ind}(T+B) = -\operatorname{ind} S$$

But $\operatorname{ind} S + \operatorname{ind} T = \operatorname{ind}(I-F_1) = 0$, so $\operatorname{ind} T = -\operatorname{ind} S$. Hence $\operatorname{ind}(T+B) = \operatorname{ind} T$.

Recall $ST = I_{\mathfrak{X}_0}$, so $S(T+B) = I+SB$ on $\mathfrak{X}_0$

Now $I+SB$ is one-to-one on $\mathfrak{X}_0$, so $\eta(T+B) \cap \mathfrak{X}_0 = \{0\}$
Since $\mathfrak{X} = \eta(T) \oplus \mathfrak{X}_0$, we must have

$$\alpha(T+B) \leq \alpha(T)$$

For suppose $n > \alpha(T)$. Let $x_1, \ldots, x_n$ be any vectors in $\eta(T+B)$. We can write

$$x_i = n_i + y_i \quad n_i \in \eta(T),\ y_i \in \mathfrak{X}_0$$

Since $\{n_1, \dots, n_n\}$ are linearly dependent, $\exists\, (\alpha_i) \neq 0$ s.t.

$$\eta(T+B) \ni \sum \alpha_i x_i = 0 + \sum \alpha_i y_i \in \mathcal{X}_0$$

$$\Rightarrow \sum \alpha_i x_i = 0$$

$$\Rightarrow \{x_1, \dots, x_n\} \text{ lin. dep}$$

$$\Rightarrow \dim \eta(T+B) < n$$

Since n was any number bigger than $\alpha(T)$, we have $\alpha(T+B) \leq \alpha(T)$.
Argument for $\beta(T+B)$ is similar

$\square$

7.4.15 <u>THEOREM:</u> Let $T \in B(\mathcal{X}, \mathcal{Y})$, $S \in B(\mathcal{Y}, \mathcal{Z})$. If $ST \in \Phi(\mathcal{X}, \mathcal{Z})$, then $T \in \Phi \Leftrightarrow S \in \Phi$

Proof. Suppose $T \in \Phi$. $\exists\, T_1 \in \Phi$ s.t.

$$TT_1 = I - F_1 \Rightarrow STT_1 = S - SF_1$$

Then $S = (ST)T_1 + SF_1$ = Fredholm + finite rank = Fredholm
↑ by 7.4.12

Use other side for converse

$\square$

(Can use this to prove theorem 7.4.14)

7.4.16 THEOREM: Let $T \in B(\mathcal{X}, \mathcal{Y})$, $S \in B(\mathcal{Y}, \mathcal{Z})$ s.t $ST \in \Phi(\mathcal{X}, \mathcal{Z})$

(a) If $\alpha(S) < \infty$, then $T, S \in \Phi$

(b) If $\beta(T) < \infty$, then $T, S \in \Phi$

Proof. (a) $\mathcal{R}(S) \supset \mathcal{R}(ST)$. Since $\beta(ST) < \infty$, then $\mathcal{R}(S)$ is closed and $\beta(S) \leq \beta(ST)$. Hence S is fredholm, so by 7.4.15, T is also Fredholm

(b) For (b) use adjoints and part (a) $[\alpha(T^*) = \beta(T) < \infty]$

7.4.17 Example: $T : \ell_2(\mathbb{N}) \longrightarrow \ell_2(\mathbb{Z})$

$$T(x_1, x_2, \dots) = (\dots 0, (0), x_1, x_2, \dots)$$

$$\alpha(T) = 0 \quad \beta(T) = \infty$$

$\mathcal{R}(T)$ closed

$$\Rightarrow T \in \Phi_+$$

Let $S : \ell_2(\mathbb{Z}) \longrightarrow \ell_2(\mathbb{N})$ by $S(\dots 0, (0), x_1, x_2 \dots) = (x_1, x_2, \dots)$
Then $\alpha(S) = \infty$, $\beta(S) = 0$ $\Rightarrow S \in \Phi_-$

However, $ST = I \in \Phi$.

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§8.1 The Single Valued Extension Property (SVEP)

$T \in B(\mathcal{X})$, $\mathcal{X}$ complex B-space

8.1.1. DEFINITION: $T \in B(\mathcal{X})$ has SVEP at $\lambda_0 \in \mathbb{C}$ if for any analytic function $f : N_{\lambda_0} \to \mathcal{X}$ (N_{λ_0} nbhd of λ_0) such that if

$$(\lambda I - T) f(\lambda) = 0 \quad \forall \lambda \in N_{\lambda_0}$$

then $f(\lambda) = 0 \quad \forall \lambda \in N_{\lambda_0}$.

T has the SVEP if it has SVEP at every $\lambda_0 \in \mathbb{C}$

(Write $T \in \mathcal{P}(\mathcal{X})$)

Suppose λ_0 is in resolvent. Then T has SVEP at λ_0, since

$$f(\lambda) = R(T;\lambda)(\lambda I - T) f(\lambda) = 0$$

8.1.2. THEOREM: If T fails SVEP, then $\sigma_p(T)$ has a non-empty open set.

Proof. $\exists$ nbhd N_{λ_0} and $f \neq 0$ with $(\lambda - T) f(\lambda) = 0$ for every $\lambda \in N_{\lambda_0}$. If $\lambda_1 \in N_{\lambda_0}$, $f(\lambda_1) \neq 0$, then $\lambda_1 \in \sigma_p(T)$

If $f(\lambda_1)=0$, then

$$(\lambda_1 - T)\, f'(\lambda) + f(\lambda) = \left[(\lambda - T) f(\lambda)\right]' = 0$$

$$\Rightarrow (\lambda_1 - T)\, f'(\lambda_1) = 0$$

If $f'(\lambda_1) \neq 0$, then $\lambda_1 \in \sigma_p(T)$. If $f'(\lambda_1) = 0$, then

$$(\lambda - T)\, f'(\lambda) + f'(\lambda) + f'(\lambda) = 0$$

$$\Rightarrow (\lambda_1 - T)\, f''(\lambda_1) = 0$$

If $f''(\lambda_1) \neq 0$, then $\lambda_1 \in \sigma_p(T)$. If $f''(\lambda_1) = 0$, proceed as before. Eventually $f^{(n)}(\lambda_1) \neq 0$ since f is analytic.

▨

(Actually showed $N_{\lambda_0} \subset \sigma_p(T)$)

8.1.3 Remark - Converse of theorem 8.1.2 is not true. Let $\mathcal{X}$ be a non-separable Hilbert space. Let $\{e_\lambda : |\lambda| \leq 1\}$ be a complete orthonormal set for $\mathcal{X}$. Define

$$Te_\lambda := \lambda e_\lambda$$

Then $\{|\lambda| \leq 1\} \subset \sigma_p(T)$. Note that $T^* e_\lambda = \lambda^* e_\lambda$. T is normal therefore. Suppose $(\lambda - T) f(\lambda) = 0$ for all λ in some nbhd of 0 $N_0 \subset \{|\lambda| \leq 1\}$. Then

(eigenvectors for distinct eigenvalues)

$$f(\lambda_1) \perp f(\lambda_2) \quad \text{for } \lambda_1 \neq \lambda_2$$

Hence $f(\lambda) = c_\lambda e_\lambda$, but this is not continuous unless $f(\lambda) = 0 \ \forall \lambda \in N_0$
Hence T has SVEP

8.1.4 THEOREM: The following operators have SVEP

(a) All operators on finite dimensional spaces
(b) All compact operators
(c) T with $\sigma(T)$ or $\sigma_p(T)$ nowhere dense
(d) Hermitian operators
(e) Unitary operators
(f) Normal operators

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8.1.5. THEOREM: Let $T \in \mathcal{B}(\mathfrak{X})$ be surjective but not injective. Then T fails SVEP.

Proof. Will show T fails SVEP at $\lambda = 0$. By the Open Mapping theorem, $\exists K > 0$ s.t. T maps $\{\|x\| \leq K\}$ onto the unit ball. T not 1-1 implies $\exists x_0$, $\|x_0\| = 1$ s.t. $Tx_0 = 0$. T surjective, so $\exists x_1$, $\|x_1\| \leq K$ s.t. $Tx_1 = x_0$. By induction $\exists (x_n) \subset \mathfrak{X}$ with

$$\|x_{n+1}\| \leq K \|x_n\|$$
$$Tx_{n+1} = x_n$$

Note $\|x_n\| \leq K^n$. Define $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n x_n$. This converges for $|\lambda| < 1/K$. Now $f(0) = x_0 \neq 0$, so $f \not\equiv 0$ on $|\lambda| < 1/K$. However,

$$(\lambda - T) \sum_0^N \lambda^n x_n = \lambda^{N+1} x_N$$

$$\Rightarrow \left\| (\lambda - T) \sum_0^N \lambda^n x_n \right\| < |\lambda|^{N+1} K^N = |\lambda|\, |\lambda K|^N \to 0$$

(since $\lambda K < 1$)

$$\Rightarrow (\lambda - T) f(\lambda) = 0 \quad \forall\, |\lambda| < 1/K$$

Therefore T fails SVEP near $\lambda = 0$. ∎

8.1.6. Remark (a) The above condition for the failure of SVEP is not necessary

(b) If T fails SVEP at $\lambda = 0$, then $a-T$ fails SVEP at $\lambda = a$. Hence if T has SVEP, then $a-T$ possesses SVEP

8.1.7 COROLLARY: Let T have SVEP.

(a) $\lambda \in \rho(T) \iff \lambda - T$ is onto

(b) $\lambda \in \sigma(T) \iff \lambda - T$ is not onto

Proof. $\lambda \in \rho(T) \Rightarrow \lambda - T$ bijective $\Rightarrow \lambda - T$ onto. Conversely, if $\lambda - T$ is onto and T has SVEP, then $\lambda - T$ must be injective. So $\lambda \in \rho(T)$

8.1.8. Remarks

(a) If T has a right inverse S but no left inverse, then T is onto, but not 1-1 (otherwise invertible by Open Mapping Theorem) Hence T fails SVEP. In particular, if $x_0 \neq 0$, $x_0 \in \eta(T)$, then

$$f(\lambda) := \sum_{n=0}^{\infty} (S^n x_0) \lambda^n \neq 0$$

but $(\lambda - T) f(\lambda) = 0$

(b) If T is an isometry but not onto, then T^* fails SVEP. For T^* is onto but not 1-1 (since T has closed range)

8.1.9. Corollary

(a) If T has SVEP, then $\sigma_a(T^*) = \sigma(T^*) = \sigma(T)$

(b) If T^* has SVEP, then $\sigma_a(T) = \sigma(T)$

Proof: (a) Always have $\sigma_a(T^*) \subset \sigma(T^*)$. Suppose $\lambda \in \sigma(T^*) = \sigma(T)$ but $\lambda \notin \sigma_a(T^*)$. Then

$$\|(\lambda - T^*)x^*\| \geq m\|x^*\|$$

for some m, i.e. $\lambda - T^*$ is 1-1 and has closed range. Therefore, $\lambda - T$ is onto. But $\lambda \in \sigma(T)$, so $\lambda - T$ is not invertible. Hence $\lambda - T$ is not 1-1 (since it is onto) and so T fails SVEP $\lightning$

(b) If $\lambda \in \sigma(T)$, $\lambda \notin \sigma_a(T)$, then $R(\lambda - T)$ is closed and $\lambda - T$ is injective. Then $\lambda - T^*$ is onto, so $\lambda \in \rho(T^*)$ $\lightning$

$\boxtimes$

8.1.10 Remarks

(a) If T has SVEP, αT has SVEP for all scalers α

(b) The sum of two operators with SVEP need not have SVEP

(c) The set of operators with SVEP is not normed closed

(e.g. $\mathfrak{X} = \ell_2(\mathbb{Z})$ $\quad T_k e_n = \begin{cases} e_{n-1} & n \neq 0 \\ \frac{1}{k} e_{-1} & n = 0 \end{cases}$. Then $\sigma(T_k) \subset \{|\lambda| = 1\}$

$\Rightarrow T_k$ has SVEP

$T_k \to T$ where $Te_n = \begin{cases} e_{n-1} & n \neq 0 \\ 0 & n = 0 \end{cases}$, and T fails SVEP)

(d) The set of operators with SVEP is not open

(e.g. 0 has SVEP but in any nbhd of 0 there is a shift operator (which fails SVEP)).

(e) If T has SVEP and K is compact, then $T+K$ may fail SVEP. If T fails SVEP and K is compact, then $T+K$ may have SVEP (see example in (c))

(f) If Q is quasi-nilpotent, then either both T and $T+Q$ have SVEP, or both fail SVEP

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(g) If $f \in \mathcal{F}(T)$ and f is non-constant on every nbhd of $\sigma(T)$ then $f(T)$ has SVEP iff T has SVEP

(h) The set of operators without SVEP has a non-empty open set

8.1.12 <u>THEOREM</u> (Vasilescu) Let (T_n) has SVEP. Suppose $T_n \to T$ in $B(\mathcal{X})$ and $T_n T = T T_n$ $\forall n$. Then T has SVEP.

8.1.13. <u>THEOREM.</u> Let $T_j \in B(\mathcal{X}_j)$ $j=1,2$. Then $T_1 \oplus T_2$ has SVEP iff each T_i has SVEP

[See Erdelyi - Lange, pp 9+10]

§8.2 The Local Spectrum

Let $T \in B(\mathcal{X})$, $x \in \mathcal{X}$. The function $\lambda \xmapsto{f} R(\lambda;T)x$ is analytic for $\lambda \in \rho(T)$. Then

$$(*) \qquad (\lambda - T) f(x) = x \quad \forall \lambda \in \rho(T)$$

If f is defined on some some $\Omega \subset \mathbb{C}$ with $\rho(T) \subset \Omega$ and which satisfies (*), we call f an analytic extension to Ω of $R(\cdot\,;T)x$.
$\forall \lambda \in \Omega$

If T has SVEP, then any two analytic extensions of $R(\cdot\,;T)x$ must agree on the intersection of their domain, for

$$(\lambda - T)(f(\lambda) - g(\lambda)) = 0 \quad \forall \lambda \in \Omega_f \cap \Omega_g$$

8.2.1. **Lemma**: If T has SVEP and $x \in \mathfrak{X}$, then there exists a unique analytic extension of $R(\cdot\,;T)x$ whose domain is maximal (under set inclusion)

[$(f_\gamma, \Omega_\gamma)$ ext., let $\Omega := \bigcup \Omega_\gamma$]

8.2.3. **Definition**: If T has SVEP and $x \in \mathfrak{X}$, denote the unique maximal analytic extension of $R(\cdot\,;T)x$ by

a) $\tilde{x}(\cdot)$
or b) $\tilde{x}_T(\cdot)$
or c) $\tilde{x}(\cdot\,;T)$

and call it the local resolvent for x. Denote the (open) domain of $\tilde{x}$ by $\rho_T(x)$ or $\rho(x;T)$ and call this domain the local resolvent set at x. Let $\sigma_T(x)$ or $\sigma(x;T)$ be the complement of the local resolvent set. This is called the local spectrum for x. Note that $\sigma_T(x) \subset \sigma(T)$ is closed and compact.

8.2.4 Examples

(a) $x = 0$. $\quad R(\lambda;T)x = 0 \ \forall\lambda$ so $\tilde{x}(\lambda) = 0 \ \forall \lambda \in \mathbb{C}$

$$\rho_T(x) = \mathbb{C}$$
$$\sigma_T(x) = \emptyset$$

(b) Let $Tx_0 = \lambda x_0$. $\quad$ Let $\tilde{x}_0(\lambda) := \frac{1}{\lambda - \lambda_0} x_0$, $\lambda \neq \lambda_0$.

Note that

$$(\lambda - T)\tilde{x}_0(\lambda) = \frac{1}{\lambda - \lambda_0}(\lambda - T)(x_0)$$

$$= \frac{1}{\lambda - \lambda_0}\left[(\lambda - \lambda_0)x_0 + \underbrace{\cancel{(\lambda_0 - T)x_0}}_{0}\right]$$

$$= x_0$$

for every $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$. Hence $\sigma_T(x_0) = \{\lambda_0\}$ (As we will see later)

(c) Let $\mathfrak{X} = C[0,1]$. Let $(Tx)(t) := t\,x(t)$.

Claim: $\sigma(T) = [0,1] = \sigma_r(T)$. $\quad R(\lambda;T)x(t) = \frac{1}{\lambda - t}x(t)$

[So $\sigma_p(T)$ is empty $\Rightarrow$ T has SVEP] $\quad \forall \lambda \notin [0,1]$

Range of $\lambda - T$ are functions which vanish at λ. This is not dense in $C[0,1]$ so $\sigma(T) = \sigma_r(T)$.

Let $x_0(t) = 0$ for $1/2 \le t \le 1$. If $1/2 < \lambda \le 1$, then take

$$\tilde{x}_0(t) := \frac{1}{\lambda - t} x_0(t) \in C[0,1]$$

$$\uparrow \; x_0(\lambda) = 0$$

Note $\sigma(x_0) \subset [0, 1/2]$

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8.2.5 DEFINITION: If $H \subset \mathbb{C}$, define

$$\mathfrak{X}_T(H) := \{ x \in \mathfrak{X} : \exists f_x \text{ analytic on } \mathbb{C} \setminus H \text{ s.t. } (\lambda - T) f_x(\lambda) = x \ \forall \lambda \in \mathbb{C} \setminus H \}$$

If T has SVEP, then $\mathfrak{X}_T(H) = \{ x \in \mathfrak{X} : \sigma_T(x) \subset H \}$. Call $\mathfrak{X}_T(H)$ the spectral manifold corresponding to H.

8.2.6 THEOREM: Let T have SVEP

(a) If $a \neq 0$, then $\sigma_T(ax) = \sigma_T(x)$

(b) $\sigma_T(x+y) \subset \sigma_T(x) + \sigma_T(y)$

(c) $\sigma_T(x) = \emptyset \iff x = 0$

(d) $AT = TA \Rightarrow \sigma_T(Ax) \subset \sigma_T(x)$

(e) If $\lambda \in \rho_T(x)$, then $\sigma_T(\tilde{x}_T(\lambda)) = \sigma_T(x)$

(f) $\sigma(T) = \bigcup_{x \in \mathfrak{X}} \sigma_T(x)$

Proof. (a) $(\lambda - T) a\tilde{x}(\lambda) = ax \quad \forall \lambda \in \rho_T(x)$. Hence $\rho_T(x) \subset \rho_T(ax)$ and so $\sigma_T(ax) \subset \sigma_T(x)$. But then

$$\sigma_T\left(\tfrac{1}{a}(ax)\right) \subset \sigma_T(ax) \subset \sigma_T(x)$$
$$\| $$
$$\sigma_T(x)$$

(b) Let $\lambda \in \rho_T(x) \cap \rho_T(y)$. Then

$$(\lambda - T)\left(\tilde{x}(\lambda) + \tilde{y}(\lambda)\right) = x + y$$

$\tilde{x}(\cdot) + \tilde{y}(\cdot) \subset \widetilde{x+y}(\cdot)$, so $\rho_T(x) \cap \rho_T(y) \subset \rho_T(x+y)$. Hence

$$\sigma_T(x+y) \subset \sigma_T(x) + \sigma_T(y)$$

(c) $x=0$ clear

Conversely, let $\sigma(x) = \phi$. Then $\tilde{x}(\cdot)$ is defined for all $\lambda \in \mathbb{C}$. If $|\lambda| > \|T\|$, $\tilde{x}(\lambda) = R(\lambda;T)x \to 0$ as $|\lambda| \to \infty$. Hence $\tilde{x}(\cdot)$ is entire and bounded, so it must be constant. Since $\tilde{x}(\lambda) \to 0$, $\tilde{x}(\lambda) = 0 \ \forall \lambda$. Therefore

$$x = (\lambda - T)\tilde{x}(\lambda) = 0$$

(d) $(\lambda - T)\tilde{x}(\lambda) = x \quad \forall \lambda \in \rho_T(x)$

$$\Rightarrow A(\lambda - T)\tilde{x}(\lambda) = Ax \quad \forall \lambda \in \rho_T(x)$$

$$\Rightarrow (\lambda - T)A\tilde{x}(\lambda) = Ax \quad \forall \lambda \in \rho_T(x)$$

$$\Rightarrow A\tilde{x}(\cdot) \subset \widetilde{Ax}(\cdot)$$

$$\Rightarrow \rho_T(x) \subset \rho_T(Ax)$$

$$\Rightarrow \sigma_T(Ax) \subset \sigma_T(x)$$

(e) Let $\lambda_0 \in \rho_T(x)$. Define

$$g(\lambda) := \begin{cases} -\dfrac{\tilde{x}(\lambda) - \tilde{x}(\lambda_0)}{\lambda - \lambda_0} & \lambda \in \rho_T(x),\ \lambda \neq \lambda_0 \\ -\tilde{x}'(\lambda_0) & \lambda = \lambda_0 \end{cases}$$

Then g is analytic in $\rho_T(x)$

$$(\lambda - T)g(\lambda) = \frac{x - (\lambda - \lambda_0)\tilde{x}(\lambda_0) - x}{-(\lambda - \lambda_0)} = \tilde{x}(\lambda_0)$$

Hence

$$(\lambda_0 - T)g(\lambda_0) = \tilde{x}(\lambda_0)$$

$$\Rightarrow g(\cdot) = \widetilde{\tilde{x}(\lambda_0)}(\cdot)$$

$$\Rightarrow \rho(x) \subset \rho(\tilde{x}(\lambda_0))$$

$$\Rightarrow \sigma(\tilde{x}(\lambda_0)) \subset \sigma(x)$$

Let $z = \tilde{x}(\lambda_0)$.

$$(\lambda - T)\,\tilde{z}(\lambda) = z = \tilde{x}(\lambda_0) \quad \forall \lambda \in \rho_T(z)$$

$$= (\lambda_0 - T)(\lambda - T)\tilde{z}(x) = (\lambda_0 - T) z = x$$

$$\Rightarrow (\lambda - T)\left[(\lambda_0 - T)\tilde{z}(\lambda)\right] = x$$

$$\Rightarrow (\lambda_0 - T)\tilde{z}(\cdot) \subset \tilde{x}$$

$$\Rightarrow \rho_T(z) \subset \rho_T(x)$$

$$\Rightarrow \rho_T(x) \subset \rho_T(z) = \rho_T(\tilde{x}(\lambda_0))$$

(f) $\sigma_T(x) \subset \sigma(T)\ \forall x \Rightarrow \bigcup \sigma_T(x) \subset \sigma(T)$. Let $\lambda_0 \notin \bigcup \sigma_T(x)$. Then $\lambda_0 \notin \sigma_T(x)\ \forall x$, so $\lambda_0 \in \rho_T(x)\ \forall x \in \mathfrak{X}$. Hence $\lambda_0 - T$ is onto for

$$(\lambda_0 - T)\tilde{x}_T(\lambda_0) = x$$

But $\lambda_0 - T$ has SVEP, so $\lambda_0 - T$ is invertible. Therefore $\lambda_0 \in \rho(T)$ i.e. $\lambda_0 \notin \sigma(T)$. Hence $\sigma(T) \subset \bigcup \sigma_T(x)$

$\blacksquare$

8.2.7. **THEOREM:** Let T have SVEP, $H \subset \mathbb{C}$.

(a) $\mathcal{X}_T(H)$ is a linear manifold in $\mathcal{X}$ and is T-hyperinvariant (it is invariant under any A with $AT = TA$)

(b) $\overline{\mathcal{X}_T(H)}$ is a subspace and is T-hyperinvariant

(c) $\mathcal{X}_T(H)^{\perp} = \overline{\mathcal{X}_T(H)}^{\perp}$ is T^*-hyperinvariant.

Proof. All follows from previous theorem.

8.2.8 **THEOREM:** Let T have SVEP. If H_α is a collection of sets in $\mathbb{C}$, then

$$\mathcal{X}_T\left(\bigcap_\alpha H_\alpha\right) = \bigcap_\alpha \mathcal{X}_T(H_\alpha)$$

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Note - $\mathfrak{X}_T(H) = \{x \in \mathfrak{X} : \sigma_T(x) \subset H\}$ may not be closed.

8.2.9. THEOREM: Let T have SVEP. Let C be a SCROC around $\sigma_T(x)$. Then for any polynomial p

$$p(T)x = \frac{1}{2\pi i} \int_C \tilde{x}(\lambda)\, p(\lambda)\, d\lambda$$

Proof Let C' be circle of diameter $\|T\|+1$. Then

$$p(T)x = \frac{1}{2\pi i} \int_{C'} R(\lambda;T)x\, p(\lambda)\, d\lambda$$

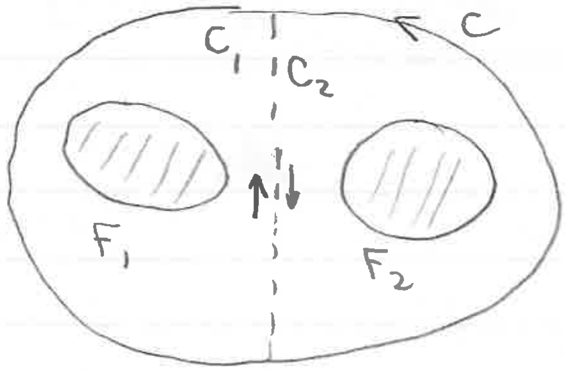
$\updownarrow$ equal on C'

$$\Rightarrow \quad p(T)x = \frac{1}{2\pi i} \int_{C'} \tilde{x}(\lambda)\, p(\lambda)\, d\lambda$$

$$\Rightarrow \quad p(T)x = \frac{1}{2\pi i} \int_{C} \tilde{x}(\lambda)\, p(\lambda)\, d\lambda \qquad (C \sim C')$$

8.2.10 THEOREM: Let T has SVEP and let $\sigma_T(x) \subset F_1 \cup F_2$ where F_1, F_2 are closed and $F_1 \cap F_2 = \emptyset$. Then x has a unique decomposition $x = x_1 + x_2$ with $\sigma_T(x_i) \subset F_i$.

Proof.



$$x = \frac{1}{2\pi i}\int_C \tilde{x}(\lambda)\,d\lambda = \frac{1}{2\pi i}\int_{C_1} \tilde{x}(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{C_2} \tilde{x}(\lambda)\,d\lambda$$

$$= x_1 + x_2$$

Show $\sigma_T(x_1)$ is inside C_1 ($\Longrightarrow \sigma_T(x_1) \subset F_1$ since C_1 arbitrary)
Let μ be outside C_1.

Claim: $\mu \in \rho_T(x_1)$. Define

$$g(\mu) := \frac{1}{2\pi i}\int_{C_1} \frac{\tilde{x}(\lambda)\,d\lambda}{\mu - \lambda}$$

Then

$$(\mu - T)\,g(\mu) = \frac{1}{2\pi i}\int_{C_1} \frac{(\mu - T)\tilde{x}(\lambda)}{\mu - \lambda}\,d\lambda$$

$$= \frac{1}{2\pi i}\int_{C_1} \frac{(\mu - \lambda)\tilde{x}(\lambda) + x}{\mu - \lambda}\,d\lambda$$

$$= \frac{1}{2\pi i}\int_{C_1} \tilde{x}(\lambda)\, d\lambda + \frac{1}{2\pi i}\left(\cancelto{0}{\int_{C_1} \frac{1}{\mu-\lambda}\, d\lambda}\right) x$$

$$= x_1 + 0 = x_1$$

This shows the claim.

For uniqueness, if $x = x_1 + x_2 = y_1 + y_2$ with $\sigma_i(y_i) \subset F_i$, then

$$x_1 - y_1 = y_2 - x_2$$

$$\sigma(x_1 - y_1) \subset F_1 \quad \sigma(y_2 - x_2) \subset F_2$$

$$\Rightarrow \sigma(x_1 - y_1) \subset F_1 \cap F_2 = \emptyset$$

$$\Rightarrow x_1 - y_1 = 0$$

$\blacksquare$

§8.3 Restrictions and Quotients

Suppose $\mathcal{Y}$ is T-invariant $(T(\mathcal{Y}) \subset \mathcal{Y})$. We regard $T|\mathcal{Y}$ as in $\mathcal{B}(\mathcal{Y})$.

$$\|T|\mathcal{Y}\| \le \|T\|$$

Also, T induces an operator in $\mathcal{X}/\mathcal{Y}$ - $T^{\mathcal{Y}} : \mathcal{X}/\mathcal{Y} \to \mathcal{X}/\mathcal{Y}$

$$T^{\mathcal{Y}}(x+\mathcal{Y}) := Tx + \mathcal{Y}$$

Then $\|T^{\mathcal{Y}}\| \le \|T\|$. Call $T^{\mathcal{Y}}$ the operator induced in $\mathcal{X}/\mathcal{Y}$, or the coinduced operator or quotient operator of T.

8.3.1. Examples:

(a) Let $\mathcal{X} = \mathbb{C}^2$ $\quad T(\alpha, \beta) = (\alpha, 2\beta)$
$\sigma(T) = \{1, 2\}$

Let $\mathcal{Y} = \{(x, 0) : x \in \mathbb{C}\}$ $\quad \sigma(T|\mathcal{Y}) = \{1\}$

$$\sigma(T|\mathcal{Y}) \subset \sigma(T)$$

Let $\mathcal{Z} = \{(0, y) : y \in \mathbb{C}\}$ $\quad \sigma(T|\mathcal{Z}) = \{2\}$

$$T^{\mathcal{Y}} \simeq T|\mathcal{Z}$$

(b) $\mathfrak{X} = \ell_\infty(\mathbb{Z}) \qquad S(e_n) = e_{n+1} \qquad$ (Right shift)

$$\sigma(S) = \{|\lambda| = 1\}$$

Let $\mathcal{Y} = \operatorname{sp}\{e_n : n \geq 1\}$

$$\sigma(S|\mathcal{Y}) = \{|\lambda| \leq 1\}$$

so $\sigma(S|\mathcal{Y}) \not\subset \sigma(S)$ [Note $\sigma(S) \subset \sigma(S|\mathcal{Y})$]

(c) $\mathfrak{X} = \mathbb{C}^2 \oplus \ell_\infty(\mathbb{Z}) \qquad R = T + S$

$$\sigma(R) = \{|\lambda| = 1\} \cup \{a\}$$

$$\sigma(R|_{\mathcal{Y}_a \oplus \mathcal{Y}_b}) = \{|\lambda| \leq 1\}$$

Note $\sigma(R) \not\subset \sigma(R|_{\mathcal{Y}_a \oplus \mathcal{Y}_b}) \not\subset \sigma(R)$

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(d) If $\mathcal{Y} \subset \mathcal{X}$ is T-invariant and $T\mathcal{X} \subset \mathcal{Y}$, then $T^{\mathcal{Y}} = 0$
Therefore $\sigma(T^{\mathcal{Y}}) = \{0\}$

Left shift $\quad S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots) \qquad \|S\| \leq 1$

$$(\lambda - S)(x_1, x_2, \dots) = (\lambda x_1 - x_2, \lambda x_2 - x_3, \dots, \lambda x_n - x_{n+1}, \dots)$$

If $x_\lambda := (1, \lambda, \lambda^2, \dots)$ belongs to the space, then

$$Sx_\lambda = \lambda x_\lambda$$

and so x_λ is an eigenvector corresponding to λ

$$\sigma(S) = \{|\lambda| \leq 1\} \qquad \sigma_p(S) \supset \{|\lambda| < 1\}$$

Now for $\lambda \neq 0$

$$(\lambda - S)\underbrace{\Big(\frac{1}{\lambda^n}, \frac{1}{\lambda^{n-1}}, \dots, \underset{\uparrow\, n^{th} \text{ spot}}{\frac{1}{\lambda}}, 0, 0, \dots\Big)}_{R(\lambda; S)e_n} = (0, 0, \dots, 0, 1, 0, \dots) = e_n$$

Similarly, for $|\lambda| < 1$

$$(\lambda - S)(0, \dots, 0, \underset{n^{th} \text{ spot}}{-1}, -\lambda, -\lambda^2, \dots) = (0, 0, \dots, 0, 1, 0, \dots)$$

Therefore S fails SVEP (onto but not 1-1)

$$(\lambda - S)\left(\frac{1}{\lambda^n}, \dots, \frac{1}{\lambda}, 1, \lambda, \lambda^2, \dots\right) = 0$$

for all $0 < |\lambda| < 1$

Let $\mathfrak{X}_n := \{(x_1, x_2, \dots, x_n, 0, 0, \dots)\}$. $\mathfrak{X}_n$ is invariant under S. $\sigma(S|\mathfrak{X}_n) = \{0\}$ since $S|\mathfrak{X}_n$ nilpotent.

$$\mathfrak{X}/\mathfrak{X}_n = \{(*, *, \dots, *, x_{n+1}, x_{n+2}, \dots\}$$

$$S^{\mathfrak{X}_n} \approx S \qquad \sigma(S^{\mathfrak{X}_n}) = \sigma(S)$$

<u>Right Shift</u> $\quad T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$

$$\sigma_r(T) \supset \{|\lambda| \leq 1\} \qquad \sigma(T) = \{|\lambda| \leq 1\}$$

$$(\lambda - T)(0, 0, \dots, \underset{\uparrow n\text{th spot}}{\frac{1}{\lambda}}, \frac{1}{\lambda^2}, \dots) = (0, 0, \dots, 0, 1, 0, \dots)$$

T has SVEP.

Let $\mathcal{Y}_n = \{(0, 0, \dots, 0, x_n, x_{n+1}, x_{n+2}, \dots)\}$ T invariant
$T|\mathcal{Y}_n \approx T$ so $\sigma(T|\mathcal{Y}_n) = \sigma(T)$.

$$\mathfrak{X}/\mathcal{Y}_n = \{(x_1, x_2, \dots, x_{n-1}, *, *, *, \dots\}$$

$\sigma(T^{\mathcal{Y}_n}) = \{0\}$.

BILATERAL SHIFT

$$T(\ldots, x_{-1}, (x_0), x_1, \ldots) = (\ldots, x_{-2}, (x_{-1}), x_0, \ldots)$$

$$\sigma(T) = \{|\lambda| = 1\} \Rightarrow T \text{ has SVEP}$$

Let $\mathcal{Y} = \{(\ldots 0, (x_0), x_1, \ldots)\}$ is T-invariant.

$$T|\mathcal{Y} \approx \text{right shift} \qquad \sigma(T|\mathcal{Y}) = \{|\lambda| \leq 1\}$$

$$\mathcal{X}/\mathcal{Y} = \{(\ldots, x_{-2}, x_{-1}, (*), *, *, \ldots)\}$$

$$T^{\mathcal{Y}} \approx \text{left shift} \qquad \sigma(T^{\mathcal{Y}}) = \{|\lambda| \leq 1\}$$

Note $T^{\mathcal{Y}}$ fails SVEP

MULTIPLICATION

$$\mathcal{X} = C[0,1] \qquad Tx(t) = t\,x(t)$$

$$R(\lambda, T)x(t) = \frac{1}{\lambda - t}\,x(t) \qquad \lambda \notin [0,1]$$

Hence $\sigma(T) = [0,1] = \sigma_r(T)$

$(\lambda - T)C[0,1]$ not dense if $\lambda \in [0,1]$

(If $\mathcal{X} = B[0,1]$, then $\sigma_p(T) = [0,1]$.

Let $\mathcal{Y} = \{x \in C[0,1] : x(t) = 0 \text{ for } 1/2 \leq t \leq 1\}$. $\mathcal{Y}$ is T-invariant.

$$\sigma(T|\mathcal{Y}) = [0, 1/2]$$

$$\mathcal{X}/\mathcal{Y} \simeq C[1/2, 1] \qquad \sigma(T|\mathcal{Y}) = [1/2, 1]$$

8.3.2 THEOREM: Let $T \in B(\mathcal{X})$, $\mathcal{Y}$ T-invariant. Then

(a) $\sigma(T) \subseteq \sigma(T|\mathcal{Y}) \cup \sigma(T^{\mathcal{Y}})$

(b) $\sigma(T|\mathcal{Y}) \subseteq \sigma(T) \cup \sigma(T^{\mathcal{Y}})$

(c) $\sigma(T^{\mathcal{Y}}) \subseteq \sigma(T) \cup \sigma(T|\mathcal{Y})$

Proof (a) $\lambda \in \rho(T|\mathcal{Y}) \cap \rho(T^{\mathcal{Y}})$. To show $\lambda \in \rho(T)$
To this end, if $(\lambda - T)x = 0$, then $(\lambda - T^{\mathcal{Y}})[x] = [0]$, and so $[x] = [0]$ since $\lambda \in \rho(T^{\mathcal{Y}})$. Hence $x \in \mathcal{Y}$. But then $(\lambda - T|\mathcal{Y})(x) = 0$ and so $x = 0$ since $\lambda \in \rho(T|\mathcal{Y})$. Hence $\lambda - T$ is 1-1.

Now if $x \in \mathcal{X}$, $\exists$ $[z]$ s.t. $(\lambda - T^{\mathcal{Y}})[z] = [x]$.
Then $(\lambda - T)z - x \in \mathcal{Y}$, and so $\exists\, w \in \mathcal{Y}$ s.t.

$$(\lambda - T)w = (\lambda - T)z - x$$

Hence $x = (\lambda - T)(z - w)$, and so $\lambda - T$ is onto
Therefore $\lambda - T$ is bijective, so $\lambda \in \rho(T)$.

$\blacksquare$

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Proof of (ii) Suppose $\lambda \in \rho(T) \cap \rho(T^{\mathcal{Y}})$. Then $\lambda - T$ is injective so $\lambda - T|_{\mathcal{Y}}$ is injective. Let $y \in \mathcal{Y}$. Then $\exists x \in \mathfrak{X}$ s.t. $y = (\lambda - T)x$. Hence

$$[0] = (\lambda - T^{\mathcal{Y}})[x]$$

But $\lambda - T^{\mathcal{Y}}$ is injective, so $[x] = [0]$, i.e. $x \in \mathcal{Y}$. Therefore $y = (\lambda - T|_{\mathcal{Y}})x$ so $\lambda - T|_{\mathcal{Y}}$ is onto. Hence $\lambda - T|_{\mathcal{Y}}$ is invertible, i.e. $\lambda \in \rho(T|_{\mathcal{Y}})$

(iii) Suppose $\lambda \in \rho(T) \cap \rho(T|_{\mathcal{Y}})$. Suppose $(\lambda - T^{\mathcal{Y}})[x] = 0$, i.e. $[(\lambda - T)x] = 0$. Then $(\lambda - T)x \in \mathcal{Y}$. Since $\lambda - T|_{\mathcal{Y}}$ is onto, $\exists$ $y \in \mathcal{Y}$ s.t.

$$(\lambda - T|_{\mathcal{Y}})\, y = (\lambda - T)x$$

But then $(\lambda - T)(y - x) = 0$ and $\lambda - T$ is 1-1, so $x = y \in \mathcal{Y}$. Hence $[x] = 0$, so $\lambda - T^{\mathcal{Y}}$ is 1-1. Now suppose $[x] \in \mathfrak{X}/\mathcal{Y}$. Then $\exists v \in \mathfrak{X}$ s.t. $(\lambda - T)v = x$, and so

$$(\lambda - T^{\mathcal{Y}})[v] = [x]$$

Therefore $\lambda - T^{\mathcal{Y}}$ is onto. Hence $\lambda \in \rho(T^{\mathcal{Y}})$

$\blacksquare$

8.3.3. **THEOREM:** Let $T \in B(\mathscr{X})$, $\mathcal{Y}$ T-invariant. TFAE for $\lambda \in \rho(T)$

(i) $\lambda \in \rho(T|\mathcal{Y})$

(ii) $R(\lambda;T)\mathcal{Y} \subset \mathcal{Y}$

(iii) $\lambda \in \rho(T^{\mathcal{Y}})$

Proof (i) $\Rightarrow$ (ii) If $\lambda \in \rho(T|\mathcal{Y})$, then $\lambda - T|\mathcal{Y}$ is 1-1 onto $\mathcal{Y}$ and so $(\lambda - T|\mathcal{Y})^{-1}$ is 1-1 of $\mathcal{Y}$ onto $\mathcal{Y}$. But

$$R(\lambda;T)|\mathcal{Y} = (\lambda - T|\mathcal{Y})^{-1}$$

and so $R(\lambda;T)\mathcal{Y} \subset \mathcal{Y}$

(ii) $\Rightarrow$ (iii) If $R(\lambda;T)\mathcal{Y} \subset \mathcal{Y}$, then $\lambda \in \rho(T|\mathcal{Y})$ and so

$R(\lambda;T|\mathcal{Y}) = R(\lambda;T)|\mathcal{Y}$

$$\lambda \in \rho(T|\mathcal{Y}) \cap \rho(T) \subset \rho(T^{\mathcal{Y}})$$

by (iii) of theorem 8.3.2.

(iii) $\Rightarrow$ (i). Then $\lambda \in \rho(T^{\mathcal{Y}}) \cap \rho(T) \subset \rho(T|\mathcal{Y})$.

$\boxtimes$

Let $\rho_\infty(T)$ be the unbounded component of $\rho(T)$. Let

$$\sigma_\infty(T) := \mathbb{C} \setminus \rho_\infty(T)$$

("full spectrum" - fill in holes of $\sigma(T)$)

8.3.4. <u>COROLLARY</u>: If $\mathcal{Y}$ is T-invariant, then $\sigma(T|\mathcal{Y}) \subset \sigma_\infty(T)$ and $R(\lambda; T|\mathcal{Y}) = R(\lambda;T)|\mathcal{Y}$ for $\lambda \in \rho(T|\mathcal{Y}) \cap \rho(T)$

Proof. Let $y \in \mathcal{Y}$. If $|\lambda| > \|T\|$, then

$$R(\lambda;T)y = \sum_{n=0}^{\infty} \frac{T^n y}{\lambda^{n+1}} \subset \mathcal{Y}$$

Claim: $R(\lambda;T)y \in \mathcal{Y}$ for all $\lambda \in \rho_\infty(T)$. Let $x^* \in \mathfrak{X}^*$ be such that

$$x^* \mathcal{Y} = 0$$

Then $x^* R(\lambda;T)y = 0 \quad \forall \lambda > \|T\|$. Hence $x^* R(\lambda;T)y = 0 \quad \forall \lambda \in \rho_\infty(T)$ since $\rho_\infty(T)$ is connected. By Hahn-Banach theorem,

$$R(\lambda;T)y \in \mathcal{Y} \quad \forall \lambda \in \rho_\infty(T)$$

(since above holds for all $x^* \in \mathfrak{X}^*$ which vanishes on $\mathcal{Y}$). Hence

$$R(\lambda;T)\mathcal{Y} \subset \mathcal{Y}$$

and so $\lambda \in \rho(T|\mathcal{Y})$, i.e. $\rho_\infty(T) \subset \rho(T|\mathcal{Y})$. ☒

8.3.5. THEOREM (Scoggs) Let $\mathcal{Y}$ be T-invariant. If G is a bounded component of $\rho(T)$, then either

$$G \cap \rho(T|\mathcal{Y}) = 0$$

$$\text{or} \quad G \subset \rho(T|\mathcal{Y})$$

Proof If $G \cap \rho(T|\mathcal{Y}) \neq \phi$, choose $\lambda_1 \in G \cap \rho(T|\mathcal{Y})$. If $G \not\subset \rho(T|\mathcal{Y})$, $\exists\, \lambda_2 \in G \setminus \rho(T|\mathcal{Y})$. Since G is pathwise connected, $\exists\, \lambda_3 \in G \cap \partial\sigma(T|\mathcal{Y})$. Then $\lambda_3 \in \sigma_a(T|\mathcal{Y}) \subset \sigma_a(T) \subset \sigma(T)$. But $\lambda \in \rho(T)$ ↯. Hence $G \subset \rho(T|\mathcal{Y})$

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8.3.6 DEFINITION: (Kariotis) Let $T \in B(\mathscr{X})$ and $\mathcal{Y}$ be T-invariant. We say that $\mathcal{Y}$ is a ν-space (has the spectral inclusion property SIP) if

$$\sigma(T|\mathcal{Y}) \subset \sigma(T)$$

8.3.7. THEOREM: Let $T \in B(\mathscr{X})$, $\mathcal{Y}$ T-invariant. TFAE

(i) $\sigma(T|\mathcal{Y}) \subset \sigma(T)$

(ii) $\sigma(T^{\mathcal{Y}}) \subset \sigma(T)$

(iii) $R(\lambda;T)\mathcal{Y} \subset \mathcal{Y} \quad \forall \lambda \in \rho(T)$

(iv) $\sigma(T) = \sigma(T|\mathcal{Y}) \cup \sigma(T^{\mathcal{Y}})$

Proof. (i) $\Rightarrow$ (ii) $\sigma(T^{\mathcal{Y}}) \subset \sigma(T) \cup \sigma(T|\mathcal{Y}) \subset \sigma(T)$
(ii) $\Rightarrow$ (i) $\sigma(T|\mathcal{Y}) \subset \sigma(T) \cup \sigma(T^{\mathcal{Y}}) \subset \sigma(T)$
(i) $\Leftrightarrow$ (iii) Theorem 8.3.3.

8.3.8 COROLLARY: (a) If $\lambda \in \rho(T)$ then $\lambda \in \sigma(T^{\mathcal{Y}}) \Leftrightarrow \lambda \in \sigma(T|\mathcal{Y})$
(b) If G is a bounded component of $\rho(T)$, then either $G \cap \sigma(T^{\mathcal{Y}}) = \emptyset$ or $G \subset \sigma(T^{\mathcal{Y}})$
(c) $\sigma(T^{\mathcal{Y}}) \subset \sigma_{\infty}(T)$

8.3.9. <u>COROLLARY:</u> If $\mathcal{Y}$ is T-hyperinvariant (invariant under any operator which commutes with T), then $\mathcal{Y}$ has SIP

Proof. By 8.3.7 (iii)

8.3.10. <u>COROLLARY:</u> If $\sigma(T)$ does not separate the plane, then every T-invariant space has SIP and $\sigma(T) = \sigma_\infty(T)$

Proof. By 8.3.4 (since in this case $\rho_\infty(T) = \rho(T)$)

<u>Other examples of spaces with SIP</u> $(T \in B(\mathcal{X}))$

(a) $\mathcal{Y} = E\mathcal{X}$ where $E^2 = E$ and $TE = ET$

(b) If $F \subset \mathbb{C}$ is closed and $\mathcal{X}_T(F)$ is closed, then $\mathcal{X}_T(F)$ has SIP

(c) If $\mathcal{Y}$ is a "spectral maximal space", then $\mathcal{Y}$ has SIP

(d) If $\sigma(T|\mathcal{Y})$ is nowhere dense, then $\mathcal{Y}$ has SIP

(e) If $\sigma(T)$ is nowhere dense and does not separate the plane, then every T-invariant subspace of $\mathcal{X}$ has SIP

8.3.11. THEOREM (a) Let $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$, where $\mathcal{Y}$ and $\mathcal{Z}$ are T-invariant. Then

$$\sigma(T) = \sigma(T|\mathcal{Y}) \cup \sigma(T|\mathcal{Z})$$

$$\sigma(T^{\mathcal{Z}}) = \sigma(T|\mathcal{Y})$$

Proof. $\mathcal{Z} \simeq \mathcal{X}/\mathcal{Y}$ and $\mathcal{Y} \simeq \mathcal{X}/\mathcal{Z}$

$$T^{\mathcal{Z}}[x]_{\mathcal{Z}} = [Tx]_{\mathcal{Z}} = [Ty + Tz]_{\mathcal{Z}} = [Ty]_{\mathcal{Z}}$$

(to be continued)

§8.4 Local Spectra, restriction, and quotients

8.4.1. THEOREM: Let T has SVEP and $\mathcal{Y}$ be T-invariant.

(a) $T|\mathcal{Y}$ has SVEP in $B(\mathcal{Y})$

(b) $\sigma_T(y) \subset \sigma_{T|\mathcal{Y}}(y) \quad \forall y \in \mathcal{Y}$

(c) If $y \in \mathcal{Y}$, $\lambda \in \rho_{T|\mathcal{Y}}(y) \subset \rho_T(y)$, then

$$\tilde{y}_T(\lambda) = \tilde{y}_{T|\mathcal{Y}}(y)$$

Proof. (a) Suppose $f : \Omega \to \mathcal{Y}$ is analytic and

$$(\lambda - T|\mathcal{Y}) f(\lambda) = 0 \qquad \forall \lambda \in \Omega$$

Then $(\lambda - T) f(\lambda) = 0$, treating f with values in $\mathcal{X}$. Since T has SVEP, $f(\lambda) = 0 \ \forall \lambda \in \Omega$, and so $T|\mathcal{Y}$ has SVEP

(b) Suppose $\lambda_0 \in \rho_{T|\mathcal{Y}}(y)$. $\exists f :$ nbhd of $\lambda_0 \to \mathcal{Y}$ s.t.

$$(\lambda - T|\mathcal{Y}) f(\lambda) = y$$

$$\Rightarrow (\lambda - T) f(\lambda) = y$$

$$\Rightarrow f \subset \tilde{y}_T$$

$$\Rightarrow \lambda_0 \in \rho_T(y)$$

(c) Note that

$$y = (\lambda - T|\mathcal{Y}) \, \tilde{y}_{T|\mathcal{Y}}(y) = (\lambda - T) \, \tilde{y}_{T|\mathcal{Y}}(y)$$

and so $\tilde{y}_T(y)$ agrees with $\tilde{y}_{T|\mathcal{Y}}(y)$ on $\rho_{T|\mathcal{Y}}(y)$

$\blacksquare$

8.4.2. Example: $\sigma(T|\mathcal{Y}) < \sigma(T)$ but $\exists\, y_0 \in \mathcal{Y}$ s.t. $\sigma_T(y_0) \subsetneq \sigma_{T|\mathcal{Y}}(y_0)$

Let $\mathcal{X} = \ell_\infty(\mathbb{Z}) \oplus \ell_\infty(\mathbb{N})$

T = right shift in both subspaces

$\mathcal{Y} = \{(0,0,\dots,0,(x_0),x_1,\dots)\} \oplus \{0\}$

Then $\sigma(T) = \{|\lambda| \le 1\} = \sigma(T|\mathcal{Y})$. Let

$$y_0 = \underbrace{e_1}_{\in \ell_\infty(\mathbb{Z})} \oplus \underbrace{0}_{\in \ell_\infty(\mathbb{N})}$$

Then $\sigma_T(y_0) = \{|\lambda| = 1\}$ since

$|\lambda| < 1$, $(\lambda - T)[(\dots, \lambda^2, \lambda, (1), 0, 0, \dots) \oplus 0] = e_1 \oplus 0$

However, $\sigma_{T|\mathcal{Y}}(y_0) = \{|\lambda| \le 1\}$

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8.3.11 THEOREM

(a) Let $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$, each T-invariant. Then $\mathcal{X}/\mathcal{Z} \simeq \mathcal{Y}$ and

$$\sigma(T^{\mathcal{Z}}) = \sigma(T|\mathcal{Y})$$

(b) If $\mathcal{X} = \mathcal{Y} + \mathcal{Z}$, then $\mathcal{X}/\mathcal{Z} \simeq \mathcal{Y}/\mathcal{Y}\cap\mathcal{Z}$ and

$$\sigma(T^{\mathcal{Z}}) = \sigma\left((T|\mathcal{Y})^{\mathcal{Y}\cap\mathcal{Z}}\right) \subset \sigma_{\infty}(T|\mathcal{Y})$$

$$\subset \sigma(T)$$

Proof. (a) Define $\Theta : \mathcal{Y} \to \mathcal{X}/\mathcal{Z}$ by $\Theta y = y + \mathcal{Z}$. Now

$$T^{\mathcal{Z}} \circ \Theta = \Theta \circ T|\mathcal{Y}$$

$$\Rightarrow \Theta^{-1} T^{\mathcal{Z}} \Theta = T|\mathcal{Y}$$

Hence $T|\mathcal{Y}$ and $T^{\mathcal{Z}}$ are similar, so $\sigma(T|\mathcal{Y}) = \sigma(T^{\mathcal{Z}})$

(b) Define $\omega : \mathcal{Y}/\mathcal{Z}_1 \to \mathcal{X}/\mathcal{Z}$ by

$$\omega(y + \mathcal{Z}_1) = y + \mathcal{Z}$$

Then $\|\omega(y+\mathcal{Z}_1)\| = \|y+\mathcal{Z}\| = \inf_{z\in\mathcal{Z}} \|y+z\| \leq \inf_{z\in\mathcal{Z}_1} \|y+z\| = \|y+\mathcal{Z}_1\|$

Finally, $\omega(T|\mathcal{Y})^{\mathcal{Z}_1} = T^{\mathcal{Z}}\omega$, so $(T|\mathcal{Y})^{\mathcal{Z}_1}$ and $T^{\mathcal{Z}}$ are similar. Hence

$$\sigma(T^{\mathcal{Z}}) = \sigma((T|\mathcal{Y})^{\mathcal{Z}_1}) \underset{\substack{\uparrow \\ 8.3.8(c)}}{\subset} \sigma_{\infty}(T|\mathcal{Y}) \underset{\substack{\uparrow \\ 8.3.4}}{\subset} \sigma_{\infty}(T)$$

$\blacksquare$

8.4.3 DEFINITION (Karistis) If T has SVEP and $\mathcal{Y}$ is T-invariant, we say that $\mathcal{Y}$ is a μ-space if

$$\sigma_T(y) = \sigma_{T|\mathcal{Y}}(y) \quad \forall y \in \mathcal{Y}$$

THEOREM: A μ-space is a ν-space (has SIP)

Proof. By 8.2.6(f)

$$\sigma(T|\mathcal{Y}) = \bigcup_{y \in \mathcal{Y}} \sigma_{T|\mathcal{Y}}(y) = \bigcup_{y \in \mathcal{Y}} \sigma_T(y)$$

$$\subset \bigcup_{x \in \mathcal{X}} \sigma_T(x) = \sigma(T)$$

8.4.5. **THEOREM:** Let T have SVEP and $\mathcal{Y}$ T-invariant. TFAE

(a) $\mathcal{Y}$ is a μ-space

(b) $\sigma_{T|\mathcal{Y}}(y) = \sigma_T(y) \quad \forall y \in \mathcal{Y}$

(b') $\sigma_{T|\mathcal{Y}}(y) \subset \sigma_T(y) \quad \forall y \in \mathcal{Y}$

(c) $\{ \tilde{y}_T(\lambda) : \lambda \in \rho_T(y) \} \subset \mathcal{Y} \quad \forall y \in \mathcal{Y}$

Proof: (b' $\Rightarrow$ c) $\lambda \in \rho_T(y) \subset \rho_{T|\mathcal{Y}}(y)$. Then

$$(\lambda - T)\, \tilde{y}_T(\lambda) = y = (\lambda - T|\mathcal{Y})\, \widetilde{y_{T|\mathcal{Y}}}(\lambda)$$

$$\Rightarrow (\lambda - T)\, \tilde{y}_T(\lambda) = (\lambda - T)\, \widetilde{y_{T|\mathcal{Y}}}(\lambda) = y$$

Hence $\tilde{y}_T(\lambda) \underset{\uparrow \text{SVEP}}{=} \widetilde{y_{T|\mathcal{Y}}}(\lambda) \in \mathcal{Y} \quad \forall \lambda \in \rho_T(y)$.

(c $\Rightarrow$ b') If $\tilde{y}_T(\lambda) \in \mathcal{Y} \quad \forall \lambda \in \rho_T(y)$

$$(\lambda - T|\mathcal{Y})\, \tilde{y}_T(\lambda) = (\lambda - T)\, \tilde{y}_T(\lambda) = y$$

$$\Rightarrow \lambda \in \rho_{T|\mathcal{Y}}(y) \text{ whenever } \lambda \in \rho_T(y)$$

$$\Rightarrow \rho_T(y) \subset \rho_{T|\mathcal{Y}}(y)$$

$\boxtimes$

8.4.6 THEOREM: Let T have SVEP. If $H \subset \mathbb{C}$ is such that

$$\mathfrak{X}_T(H) = \{x \in \mathfrak{X} : \sigma_T(x) \subset H\}$$

is closed, then $\mathfrak{X}_T(H)$ is a μ-space.

Proof. $y \in \mathfrak{X}_T(H) \Leftrightarrow \sigma_T(y) \subset H$. By 8.2.6(e),

$$\sigma_T(\tilde{y}_T(\lambda)) = \sigma_T(y) \quad \forall \lambda \in \rho_T(y)$$

Hence $\tilde{y}_T(\lambda) \in \mathfrak{X}_T(H)$ $\forall \lambda \in \rho_T(y)$. By (c) of 8.4.5, $\mathfrak{X}_T(H)$ is a μ-space.

8.4.7 THEOREM: Let T have SVEP, $\mathcal{Y}$ T-invariant. If $\sigma(T|\mathcal{Y})$ is nowhere dense, then $\mathcal{Y}$ is a μ-space.

Proof. Let $y \in \mathcal{Y}$, $\lambda \in \rho_T(y)$. Since $\rho(T|\mathcal{Y})$ is dense, $\exists$ seq $(\lambda_n) \subset \rho(T|\mathcal{Y})$ s.t. $\lambda_n \to \lambda$. By 8.4.1.

$$\begin{array}{c} \tilde{y}_T(\lambda_n) = \tilde{y}_{T|\mathcal{Y}}(\lambda_n) \in \mathcal{Y} \\ \downarrow \\ \tilde{y}_T(\lambda) \in \mathcal{Y} \end{array}$$

Now apply (c) of 8.4.5.

$\boxtimes$

8.4.8 <u>THEOREM:</u> Let $T \in B(\mathfrak{X})$ and $\sigma(T)$ nowhere dense in $\mathbb{C}$
Then a T-invariant space is a μ-space iff it is a ν-space

Proof: $\sigma(T)$ nowhere dense $\Rightarrow$ T has SVEP

$$\sigma(T|\mathcal{Y}) \subset \sigma(T)$$

hence nowhere dense

8.4.9. <u>THEOREM:</u> Let $T \in B(\mathfrak{X})$ and let $\sigma(T)$ be nowhere dense, and not separating. Then every T-invariant space is a μ-space.

4/29 SPECTRAL THEORY

8.4.10 THEOREM: Let T have SVEP and $\mathcal{Y}$ T-invariant. If $T^{\mathcal{Y}}$ has SVEP, then

$$\sigma_{T^{\mathcal{Y}}}([x]) \subseteq \sigma_T(x)$$

for every $x \in \mathfrak{X}$. Moreover,

$$\widetilde{[x]}_{T^{\mathcal{Y}}}(\lambda) = [\tilde{x}_T(\lambda)]$$

for all $\lambda \in \rho_T(x)$.

Proof. We know $(\lambda - T)\tilde{x}_T(\lambda) = x$ for all $\lambda \in \rho_T(x)$. Then

$$(\lambda - T^{\mathcal{Y}})[\tilde{x}_T(\lambda)] = [x]$$

Hence $[\tilde{x}_T(\lambda)] = \widetilde{[x]}_{T^{\mathcal{Y}}}(\lambda)$ $\forall \lambda \in \rho_T(x)$ and $\rho_T(x) \subset \rho_{T^{\mathcal{Y}}}([x])$.

$\boxtimes$

Closed Spectral Manifolds

8.4.11 THEOREM: Let T have SVEP. Let $F \subset \mathbb{C}$ be closed and suppose $\mathfrak{X}_T(F) = \{x : \sigma_T(x) \subset F\}$ is closed.

(a) $\mathfrak{X}_T(F)$ is T-hyperinvariant

(b) $\sigma(T|\mathfrak{X}_T(F)) \subset \sigma(T) \cap F$

(c) If Z is T-invariant and $\sigma(T|Z) \subset \sigma(T|\mathfrak{X}_T(F))$, then $Z \subset \mathfrak{X}_T(F)$, i.e. $\mathfrak{X}_T(F)$ is "spectral maximal"

Proof. (a) If $AT = TA$ and $\sigma_T(x) \subset F$, then by 8.2.6(d),

$$\sigma_T(Ax) \subset \sigma_T(x) \subset F$$

and so $Ax \in \mathfrak{X}_T(F)$

(b) By 8.4.1(a), the operator $T_F := T|\mathfrak{X}_T(F)$ has SVEP
If $\lambda_0 \notin \sigma(T) \cap F$, then $\tilde{x}_T(\lambda_0)$ is defined for all $x \in \mathfrak{X}_T(F)$. The map $x \mapsto \tilde{x}_T(\lambda_0)$ of $\mathfrak{X}_T(F)$ into $\mathfrak{X}_T(F)$
↑ 8.2.6(e)

Claim: $\lambda - T_F : \mathfrak{X}_T(F) \to \mathfrak{X}_T(F)$ is onto

This is because $(\lambda_0 - T_F)\tilde{x}_T(\lambda_0) = x \in \mathfrak{X}_T(F)$

$$\begin{array}{c} (\lambda_0 - T_F)\tilde{x}_T(\lambda_0) \\ \shortparallel \\ (\lambda_0 - T)\tilde{x}_T(\lambda_0) \end{array}$$

Since T_F has SVEP, it is invertible, so $\lambda_0 \in \rho(T|\mathfrak{X}_F)$

(c) If $\sigma(T|Z) \subset \sigma(T|\mathfrak{X}_T(F)) \subset F \cap \sigma(T)$, then if $z \in Z$, by 8.4.1.(b)

$$\sigma_T(z) \subset \sigma_{T|Z}(\sigma) \subseteq \sigma(T|Z) \subset F$$

Hence $z \in \mathcal{X}_T(F)$. Hence $Z \subset \mathcal{X}_T(F)$.

$\boxtimes$

8.4.12 <u>DEFINITION</u>: A T-invariant subspace $\mathcal{Y}$ is T-spectral maximal if for any T-invariant Z with $\sigma(T|Z) \subset \sigma(T|\mathcal{Y})$ it follows that $Z \subset \mathcal{Y}$

8.4.13. <u>THEOREM</u>: Let T have SVEP, $\mathcal{Y}$ T-invariant, and let $F := \sigma(T|\mathcal{Y})$.

(a) $\mathcal{Y} \subset \mathcal{X}_T(F)$

(b) If $\mathcal{Y}$ is spectral maximal and if $\mathcal{X}_T(F)$ is closed, then $\mathcal{Y} = \mathcal{X}_T(F)$

Proof. (a) Let $y \in \mathcal{Y}$, so

$$\sigma_T(y) \subset \sigma_{T|\mathcal{Y}}(y) \subset \sigma(T|\mathcal{Y}) = F$$

Hence $y \in \mathcal{X}_T(F)$, i.e. $\mathcal{Y} \subset \mathcal{X}_T(F)$

(b) If $\mathcal{X}_T(F)$ is closed, by 8.4.11(b)

$$\sigma(T|\mathcal{X}_T(F)) \subset \sigma(T) \cap F \subset F = \sigma(T|\mathcal{Y})$$

and since $\mathcal{Y}$ is spectral maximal, we must have $\mathcal{X}_T(F) \subset \mathcal{Y}$. Hence by (a), $\mathcal{X}_T(F) = \mathcal{Y}$

$\blacksquare$

8.4.14 <u>COROLLARY:</u> Let T have SVEP and $\mathcal{X}_T(F)$ be closed for all closed $F \subset \mathbb{C}$. Then a T-invariant subspace $\mathcal{Y}$ is T-spectral maximal if and only if $\mathcal{Y} = \mathcal{X}_T(F)$ for some closed $F \subset \sigma(T)$

Proof. ($\Rightarrow$) 8.4.13 (b) and 8.4.11 (b)
($\Leftarrow$) 8.4.11(c)

5/1 SPECTRAL THEORY

8.4.15 <u>DEFINITION:</u> $T \in B(\mathfrak{X})$. A T-invariant subspace $\mathcal{Y}$ is T-analytically invariant if for any open $\Omega \subset \mathbb{C}$ and analytic $f: \Omega \to \mathfrak{X}$ with $(\lambda - T) f(\lambda) \in \mathcal{Y}$ for $\lambda \in \Omega$, then $f(\lambda) \in \mathcal{Y}$

Note: T has SVEP $\Leftrightarrow$ $\{0\}$ is T-analytic invariant

8.4.16 <u>THEOREM:</u> Let T have SVEP. Then every analytic invariant space is a μ-space

Proof. By 8.4.5 $\mathcal{Y}$ is a μ-space iff $\tilde{y}_T(\lambda) \in \mathcal{Y}$ for all $\lambda \in \rho_T(y)$
But if $\mathcal{Y}$ is analytic invariant,

$$(\lambda - T)\, \tilde{y}_T(\lambda) = y \in \mathcal{Y} \quad \forall \lambda \in \rho_T(y)$$

$$\Rightarrow \tilde{y}_T(\lambda) \in \mathcal{Y} \quad \forall \lambda \in \rho_T(y)$$

and so $\mathcal{Y}$ is a μ-space.

8.4.17 <u>LEMMA:</u> Let $\mathcal{Y} \subset \mathfrak{X}$, $\Omega \subset \mathbb{C}$ open, $\hat{f}: \Omega \to \mathfrak{X}/\mathcal{Y}$ analytic. If $\lambda_0 \in \Omega$, then there exists a disk $V(\lambda_0)$ and analytic function $F: V(\lambda_0) \to \mathfrak{X}$ s.t.

$$[F(\lambda)]_{\mathcal{Y}} = \hat{f}(\lambda)$$

Proof. Let $\lambda_0 \in \Omega$. Write

$$\hat{f}(\lambda) = \sum_{n=0}^{\infty} \hat{a}_n (\lambda - \lambda_0)^n$$

(where $\hat{a}_n \in \mathcal{X}/\mathcal{Y}$) which converges for $\lambda \in V(\lambda_0) = \{\lambda : |\lambda - \lambda_0| < r\}$
For $n \in \mathbb{N}$, $\exists\, a_n \in \mathcal{X}$ with $a_n \in \hat{a}_n$ and

$$\|\hat{a}_n\| \leq \|a_n\| \leq 2\|\hat{a}_n\|$$

$$\Rightarrow \quad \|\hat{a}_n\|^{1/n} \leq \|a_n\|^{1/n} \leq 2^{1/n}\|\hat{a}_n\|^{1/n}$$

$$\Rightarrow \quad \overline{\lim}\, \|a_n\|^{1/n} = \overline{\lim}\, \|\hat{a}_n\|^{1/n}$$

Let $F(\lambda) := \sum a_n (\lambda - \lambda_0)^n$, which converges for $\lambda \in V(\lambda_0)$
It's clear that

$$[F(\lambda)]_{\mathcal{Y}} = \hat{f}(\lambda)$$

▨

8.4.18 <u>COROLLARY</u>: If in addition, $\mathcal{Y}$ is T-analytically invariant and $(\lambda - T^{\mathcal{Y}})\hat{f}(\lambda) = 0$ for $\lambda \in \Omega$, then $F(\lambda) \in \mathcal{Y}$ for all $\lambda \in V$.

Proof. $[(\lambda - T)F(\lambda)] = (\lambda - T^{\mathcal{Y}})[F(\lambda)] = (\lambda - T^{\mathcal{Y}})\hat{f}(\lambda) = 0$
and so $(\lambda - T)F(\lambda) \in \mathcal{Y} \Rightarrow F(\lambda) \in \mathcal{Y}$

$\mathcal{Y}$ anal. inv.

8.4.19 THEOREM: A T-invariant subspace $\mathcal{Y}$ is analytically invariant iff $T^{\mathcal{Y}}$ has SVEP

Proof ($\Leftarrow$) Suppose $T^{\mathcal{Y}}$ has SVEP. Let $f : \Omega \to \mathcal{X}$ be analytic with $(\lambda - T) f(\lambda) \in \mathcal{Y}$. Then

$$0 = [(\lambda - T) f(\lambda)]_{\mathcal{Y}} = (\lambda - T^{\mathcal{Y}})[f(\lambda)]_{\mathcal{Y}}$$

$$\Rightarrow [f(\lambda)]_{\mathcal{Y}} = 0 \quad \text{by SVEP}$$

$$\Rightarrow f(\lambda) \in \mathcal{Y}$$

($\Rightarrow$) Let Ω be connected open set in $\mathbb{C}$, $\lambda_0 \in \Omega$. Let $\hat{f} : \Omega \to \mathcal{X}/\mathcal{Y}$ be analytic such that

$$(\lambda - T^{\mathcal{Y}})\hat{f}(\lambda) = 0 \quad \forall \lambda \in \Omega$$

By 8.4.17 $\exists F : V(\lambda_0) \to \mathcal{X}$ s.t. $[F(\lambda)] = \hat{f}(\lambda)$. By the corollary, $F(\lambda) \in \mathcal{Y}$ and so

$$\hat{f}(\lambda) = [F(\lambda)] = 0$$

for all $\lambda \in V(\lambda_0)$. Hence $\hat{f}(\lambda) = 0 \quad \forall \lambda \in \Omega$.

$\blacksquare$

8.4.20 LEMMA: Let T have SVEP, $x \in \mathfrak{X}$. Then

$$\forall \lambda_0 \in \mathbb{C}, \qquad \sigma_T((\lambda_0 - T)x) \subset \sigma_T(x) \subset \sigma_T((\lambda_0 - T)x) \cup \{\lambda_0\}$$

Proof. First inclusion easy since $\lambda_0 - T$ commutes with T. For second inclusion, let $y := (\lambda_0 - T)x$. If $\lambda \neq \lambda_0$, $\lambda \in \rho_T(y)$, let

$$g(\lambda) := \frac{1}{\lambda_0 - \lambda}\left(\tilde{y}_T(\lambda) - x\right)$$

Then

$$(\lambda - T)\, g(\lambda) = \frac{1}{\lambda_0 - \lambda}\left(y - (\lambda - T)x\right)$$

$$= \frac{1}{\lambda_0 - \lambda}\left((\lambda_0 - T)x - (\lambda - T)x\right)$$

$$= x$$

Hence $\rho_T(y) \setminus \{\lambda_0\} \subset \rho_T(x)$

$\boxed{/}$

8.4.21 THEOREM: Let T have SVEP and let $\mathfrak{X}_T(F)$ be closed, $y \in \mathfrak{X}_T(F)$. Then for any $x \in \mathfrak{X}$ s.t. $(\lambda - T)x = y$, $x \in \mathfrak{X}_T(F)$.

Proof. $\sigma(T|\mathfrak{X}_T(F)) \subset F$. Case 1: $\lambda \in F$. By the preceding lemma

$$\sigma_T(x) \subset \sigma_T(y) \cup \{\lambda\} \subset F$$

and so $x \in \mathcal{X}_T(F)$

Case 2: $\lambda \notin F$. Here $\lambda \in \rho(T | \mathcal{X}_T(F))$ and $\lambda \in \rho_T(y)$ since $\sigma_T(y) \subset F$

Then

$$(\lambda - T)\, \tilde{y}_T(\lambda) = y$$

$$\Rightarrow \quad (\lambda - T)(x - \tilde{y}_T(\lambda)) = 0$$

$$\Rightarrow \quad x - \tilde{y}_T(\lambda) = 0 \qquad (\text{since } T \text{ has SVEP})$$

$$\Rightarrow \quad \sigma_T(x) = \sigma_T(\tilde{y}_T(x)) \underset{8.2.6(e)}{=} \sigma_T(y) \subset F$$

and so $x \in \mathcal{X}_T(F)$.

$\blacksquare$

8.4.22 Corollary: If T has SVEP and if $\mathcal{X}_T(F)$ is closed, then $\mathcal{X}_T(F)$ is T-analytically invariant. Thus $\mathcal{X}_T(F)$ is a μ-space and $T^{\mathcal{X}_T(F)} =: T^F$ has SVEP..