Solving $f_{\theta}(r) = 0$ for $90 < \theta < 180$.

The function $f_{\theta}(r)$ for $120 < \theta < 180$ when k = 3 is given by

$$f3 := 1 + r\cdot\cos(\text{theta}) + r^2\cdot\cos(2\ \text{theta}) + r^3\cdot\cos(3\cdot\text{theta}) + \frac{r^4\cdot\cos(2\ \text{theta}) + r^5\cos(3\ \text{theta}) - 1 - r\cdot\cos(\text{theta})}{1 - r^2}$$

$$1 + r\cos(\theta) + r^2\cos(2\,\theta) + r^3\cos(3\,\theta) + \frac{r^4\cos(2\,\theta) + r^5\cos(3\,\theta) - 1 - r\cos(\theta)}{-r^2 + 1} \quad \textbf{(1)}$$

Maple can apply trig identities to this to write it in terms of $\sin(\theta)$ and $\cos(\theta)$.

$simplify(f3, trig)$

$$\frac{2\,r^2\sin(\theta)^2\,(2\,r\cos(\theta) + 1)}{r^2 - 1} \quad \textbf{(2)}$$

And now we just solve for when this equals 0.

$sol3 := solve(\% = 0, r)$

$$0,\ -\frac{1}{2\cos(\theta)} \quad \textbf{(3)}$$

The function $f_{\theta}(r)$ for $90 < \theta < 120$ when k = 4 is

$$f4 := 1 + r\cdot\cos(\text{theta}) + r^2\cdot\cos(2\ \text{theta}) + r^3\cdot\cos(3\cdot\text{theta}) + r^4\cdot\cos(4\cdot\text{theta}) + \frac{r^5\cdot\cos(3\ \text{theta}) + r^6\cos(4\ \text{theta}) - 1 - r\cdot\cos(\text{theta})}{1 - r^2}$$

$$1 + r\cos(\theta) + r^2\cos(2\,\theta) + r^3\cos(3\,\theta) + r^4\cos(4\,\theta) + \frac{r^5\cos(3\,\theta) + r^6\cos(4\,\theta) - 1 - r\cos(\theta)}{-r^2 + 1} \quad \textbf{(4)}$$

$simplify(f4, trig)$

$$\frac{2\,r^2\sin(\theta)^2\,(4\,r^2\cos(\theta)^2 + 2\,r\cos(\theta) - r^2 + 1)}{r^2 - 1} \quad \textbf{(5)}$$

$sol4 := solve(\% = 0, r)$

$$0,\ \frac{-\cos(\theta) + \sqrt{-3\cos(\theta)^2 + 1}}{4\cos(\theta)^2 - 1},\ -\frac{\cos(\theta) + \sqrt{-3\cos(\theta)^2 + 1}}{4\cos(\theta)^2 - 1} \quad \textbf{(6)}$$

We have to make sure that we take the correct zero where r is positive. We can check that the third solution is the one we want by graphing it. (You can also show that for $90 < \theta < 120$, the numerator in solution 3 is positive while the denominator is negative since $-\frac{1}{2} < \cos(\theta) < 0$. So multiplying the fraction by -1 will produce a positive number.)

sol4[3]

$$-\frac{\cos(\theta) + \sqrt{-3\cos(\theta)^2 + 1}}{4\cos(\theta)^2 - 1} \quad \textbf{(7)}$$

plot$\left(sol4[3], \text{theta} = \frac{\text{Pi}}{2} .. \frac{2\,\text{Pi}}{3}\right)$