Life Beyond Separable Differential Equations

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Advanced Topics: Calculus

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http://ecademy.agnesscott.edu/~lriddle/apcalculus/riceworkshop1.pdf

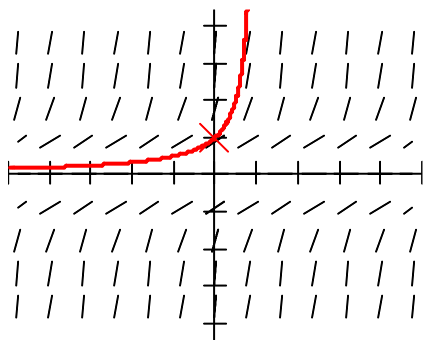
Solving Separable Differential Equations

- When solving for the general solution, have we found all solutions?
- What is the domain of a particular solution?

Example: $\frac{dy}{dx} = y^2$

By separating variables and integrating, we find the general solution is $y = \frac{-1}{x+C}$. But there is another solution, $y = 0$, which is the equilibrium solution. No value of C will give this solution. We "lost" the solution $y = 0$ when we divided by y^2 while separating variables.

Moral: When solving $\frac{dy}{dx} = g(y)h(x)$, always check first for equilibrium solutions satisfying $g(y) = 0$ before separating variables.

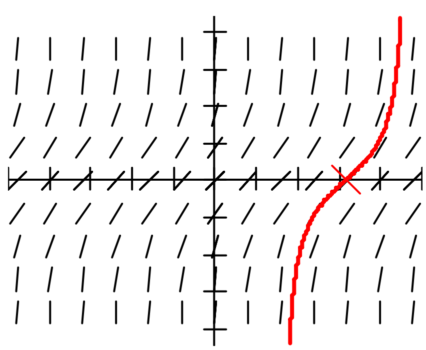


The particular solution that satisfies the initial condition $y(0) = 1$ is $y = \frac{1}{1-x}$ as sketched in the slope field to the right. The solution becomes unbounded as $x \to 1^-$ and so the solution to the initial value problem exists only for the interval $-\infty < x < 1$ even though the function $f(x) = \frac{1}{1-x}$ has domain $x \neq 1$. The solution to the initial value problem cannot jump over the vertical asymptote.

Example: $\frac{dy}{dx} = 1 + y^2, \quad y(\pi) = 0$

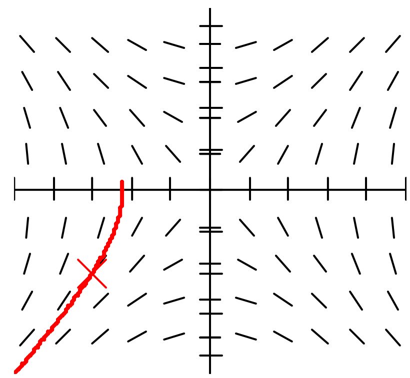
Separating variables and integrating gives $\tan^{-1}(y) = x + C$. Using the initial condition,

$$0 = \tan^{-1}(0) = \pi + C \quad \Rightarrow \quad C = -\pi \quad \Rightarrow \quad \tan^{-1}(y) = x - \pi$$



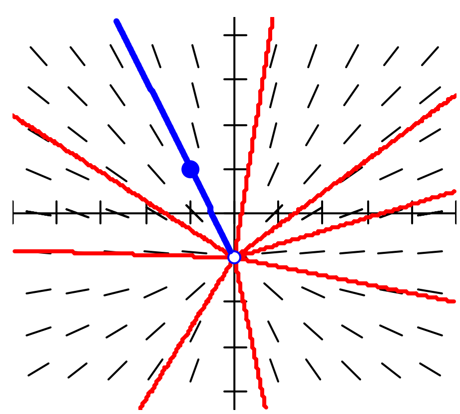
But the range of the arctangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so we must have $-\frac{\pi}{2} < x - \pi < \frac{\pi}{2}$. Thus the domain of the solution to this initial value problem is the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$, which is consistent with the sketch of the solution in the slope field. The solution can be rewritten as $y = \tan(x - \pi)$, or just $y = \tan(x)$, for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ even though the domain of the function $f(x) = \tan(x)$ is $x \neq \frac{(2n+1)\pi}{2}$, n an integer. The particular solution to the initial value problem cannot jump over either vertical asymptote.

Example: $\frac{dy}{dx} = \frac{x}{y}$, $y(-3) = -2$



Separating variables and integrating gives $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$ or $y^2 = x^2 + K$. The initial condition gives $K = -5$. Since the initial *y*-value is negative, the solution to the initial value problem is $y = -\sqrt{x^2 - 5}$. But what is the domain? This function consists of two branches of the hyperbola in the third and fourth quadrants, but the solution to the initial value problem consists only of the branch with domain $x < -\sqrt{5}$. So in solving this differential equation two choices had to be made, both based on the initial condition: one choice for which sign to take for *y* with the square root, and one choice about which interval for the domain to take to satisfy $x^2 - 5 > 0$. Also notice that $x = -\sqrt{5}$ is *not* in the domain since the differential equation is not defined when $y = 0$ and the solution is not differentiable at $x = -\sqrt{5}$.

Example (2006 AB 5): $\frac{dy}{dx} = \frac{y+1}{x}$, $x \neq 0$, with $y(-1) = 1$



The general solution for this differential equation is $y = Cx - 1$. The domain of a solution cannot contain $x = 0$, however, since no solution satisfies the differential equation when $x = 0$; it is not true that $C = \frac{0}{0}$. The slope field to the right shows several particular solutions. Each is a half-line ending or starting at $(0, -1)$. The solution in blue represents the particular solution satisfying $y(-1) = 1$. The domain of this solution is the interval $-\infty < x < 0$ even though the function $f(x) = -2x - 1$ has domain of all real *x*. The particular solution to the initial value problem cannot jump over the singularity at $x = 0$ just as the particular solutions in the first two examples cannot jump over the asymptotes. The blue curve is the unique solution to this initial value problem

Other problems to consider:

1. Investigate the solution to the differential equation $\frac{dy}{dx} = \sqrt{1-y^2}$ that satisfies $y(2)=0$, and give the interval that represents its domain. Compare with the sketch of the solution on a slope field.

2. Find the general solution to $\frac{dy}{dx} = x(y^2-1)$. Describe the domain for the particular solution with initial value $y(0)=y_0$. The domain interval may depend on the value of y_0. If possible, investigate with technology by drawing solutions on a slope field.

For similar discussion and examples, see David Lomen's article "Solving Separable Differential Equations: Antidifferentiation and Domain Are Both Needed" in the Course Home Pages section of AP Calculus at the AP Central website.

Population Models

- Exponential Growth: $\dfrac{dy}{dt} = ky$

- Logistic Growth: $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

  For small *y*, the population should grow according to the exponential growth model. For *y* greater than some carrying capacity *M* the population should decrease.

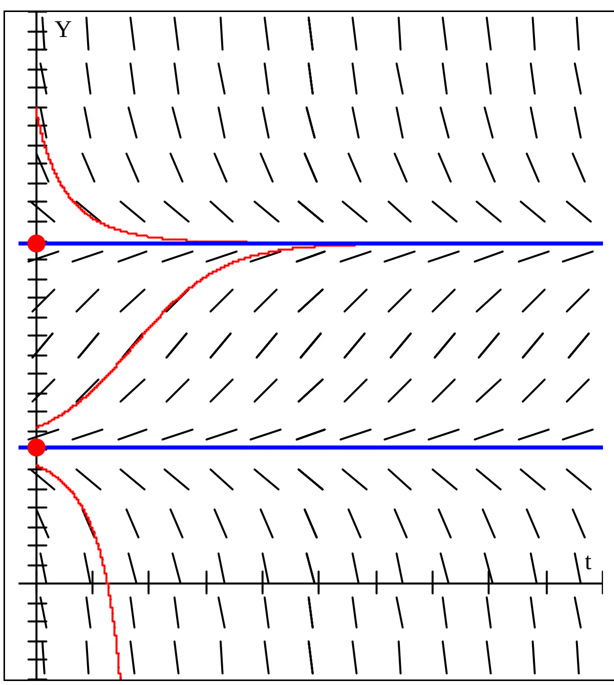
  Properties:

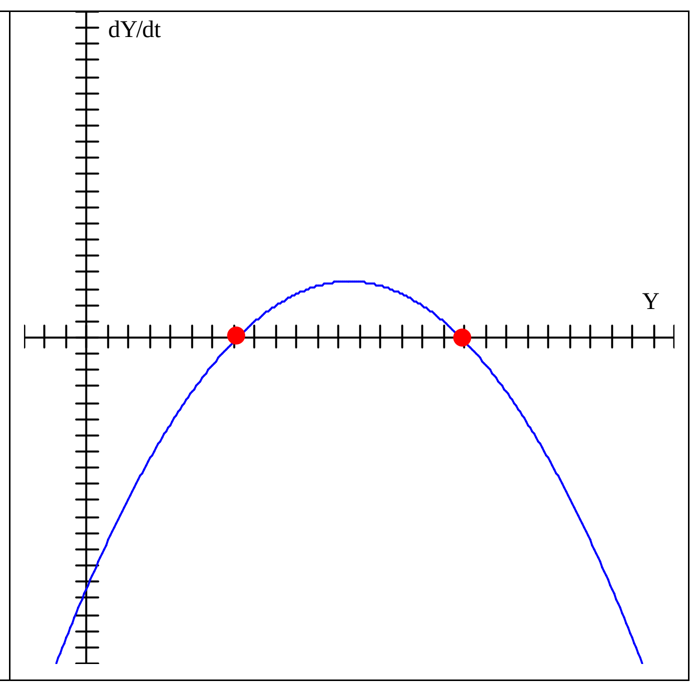
  1. Equilibrium solutions are $y = 0$ and $y = M$.
  2. If $y(0) = y_0 > 0$, then $\lim_{t\to\infty} y(t) = M$ and $\lim_{t\to\infty} y'(t) = 0$.
  3. If $y(0) = y_0$ where $0 < y_0 < \dfrac{M}{2}$, then the graph of $y(t)$ for $t > 0$ will have an inflection point when $y = \dfrac{M}{2}$.

- Logistic Growth with Constant Harvesting: $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right) - H$

  There will now be two positive equilibrium values if $0 < H < \dfrac{kM}{4}$.

  If the initial population is less than the smaller equilibrium value, the population will go extinct. If the initial population is greater than the smaller equilibrium value, the population will survive. This is illustrated in the graphs below.





If $H > \frac{kM}{4}$, the population will go extinct for any initial population size y_0. The time T to extinction is given by

$$T = \int_0^T dt = \int_{y_0}^{0} \frac{1}{ky\left(1 - \frac{y}{M}\right) - H}\, dy$$

- Predator-Prey Model (also known as Lotka-Volterra Model):

  Let $x(t)$ be the prey population at time *t*, and let $y(t)$ be the predator population at time *t* (in some appropriate units).

  Assumptions:

  1. In the absence of the predator, the prey grows at a rate proportional to the current population.
  2. In the absence of the prey, the predator dies out at a rate proportional to the current population.
  3. The number of encounters between predator and prey is proportional to the product of their populations. Each such encounter tends to promote the growth rate of the predator and to inhibit the growth rate of the prey.

  The model:

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

  Properties:

  1. Equilibrium solutions at $(0,0)$ and $\left(\frac{c}{d}, \frac{a}{b}\right)$.
  2. Solution curves in the *xy*-plane are closed curves and hence $x(t)$ and $y(t)$ are periodic. This can be proved using the following observation (the meaning of dx as the product of *d* and *x*, not as a differential, can be determined from context).

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-cy + dxy}{ax - bxy} \Rightarrow \frac{a - by}{y} dy = \frac{-c + dx}{x} dx$$

$$\Rightarrow a \ln y - by = -c \ln x + dx + C$$

$$\Rightarrow y^a e^{-by} = x^{-c} e^{dx} K$$

$$\Rightarrow \left(x^c e^{-dx}\right)\left(y^a e^{-by}\right) = K = \left({x_0}^c e^{-dx_0}\right)\left({y_0}^a e^{-by_0}\right)$$

for all points (x, y) on a solution curve, where (x_0, y_0) is a given point on the curve. (See example below.)

3. What is the average prey population? Let T be the period of a solution $x(t)$. Then

$$\begin{aligned}\bar{x} &= \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^T \frac{y'(t) + c\,y(t)}{d\,y(t)}dt \\ &= \frac{1}{Td}\int_0^T \frac{y'(t)}{y(t)}dt + \frac{c}{Td}\int_0^T dt \\ &= \frac{1}{Td}(\ln(y(T)) - \ln(y(0)) + \frac{c}{d} = \frac{c}{d}\end{aligned}$$

since $y(T) = y(0)$. Similarly, the average predator population is $\bar{y} = \frac{a}{b}$. Notice, therefore, that the average population sizes are always the same for any solution curve, and are equal to the equilibrium values.

- Competition Model

Let $x(t)$ and $y(t)$ be two populations at time *t* that compete for resources.

Assumptions:

1. Each population grows according to a logistic model in the absence of the other population.
2. The number of encounters between the two populations is proportional to the product of their populations. Each such encounter tends to inhibit the growth rate of the populations because of competition.

The model:

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x}{K_1}\right) - axy$$
$$\frac{dy}{dt} = r_2 y\left(1 - \frac{y}{K_2}\right) - bxy$$

Property:
Depending on the relative sizes of *a* and *b*, and the initial condition, one or the other species dies out and the surviving population stabilizes at its carrying capacity.

Winplot, free software for graphing solutions to differential equations
http://math.exeter.edu/rparris/

Predator-Prey (Lotka-Volterra) Model

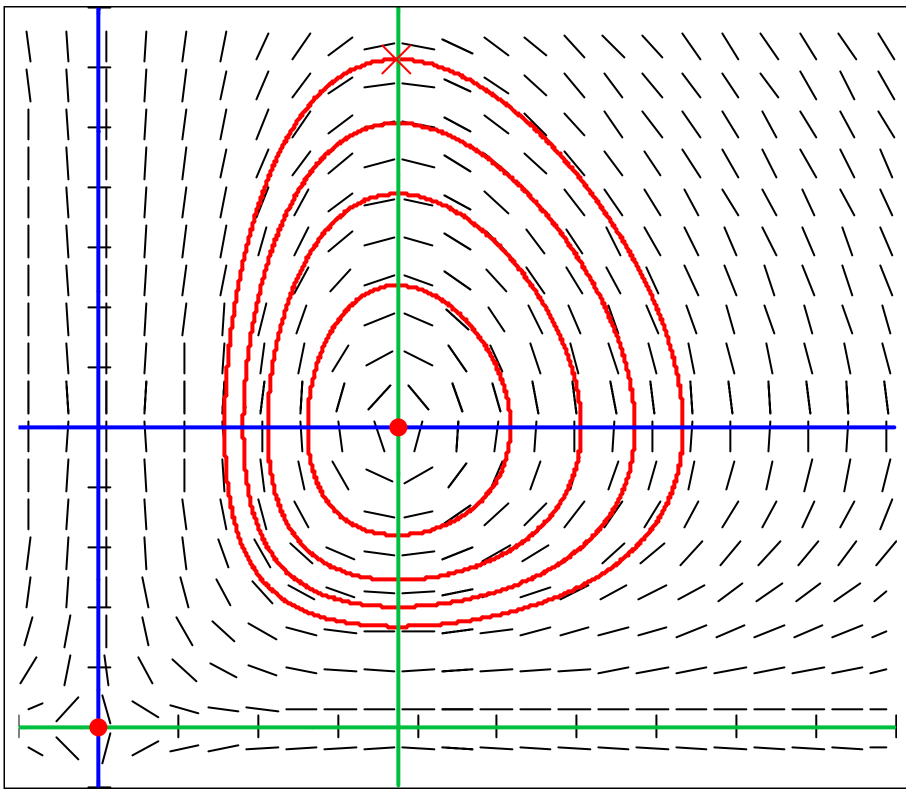
$$\frac{dx}{dt} = 2x - 0.4xy$$

$$\frac{dy}{dt} = -3y + 0.8xy$$

Let $x(t)$ be the prey population at time *t*, and let $y(t)$ be the predator population at time *t* (in some appropriate units).

Assumptions:

4. In the absence of the predator, the prey grows at a rate proportional to the current population.
5. In the absence of the prey, the predator dies out at a rate proportional to the current population.
6. The number of encounters between predator and prey is proportional to the product of their populations. Each such encounter tends to promote the growth rate of the predator and to inhibit the growth rate of the prey.



Blue Lines:
nullcline for $\frac{dx}{dt} = 0$

Green Lines:
nullcline for $\frac{dy}{dt} = 0$

Red Dots:
equilibrium points

Red Curves:
solution curves

Why are solution curves cycles in the phase plane? Consider the curve passing through the point (2,3). That curve satisfies

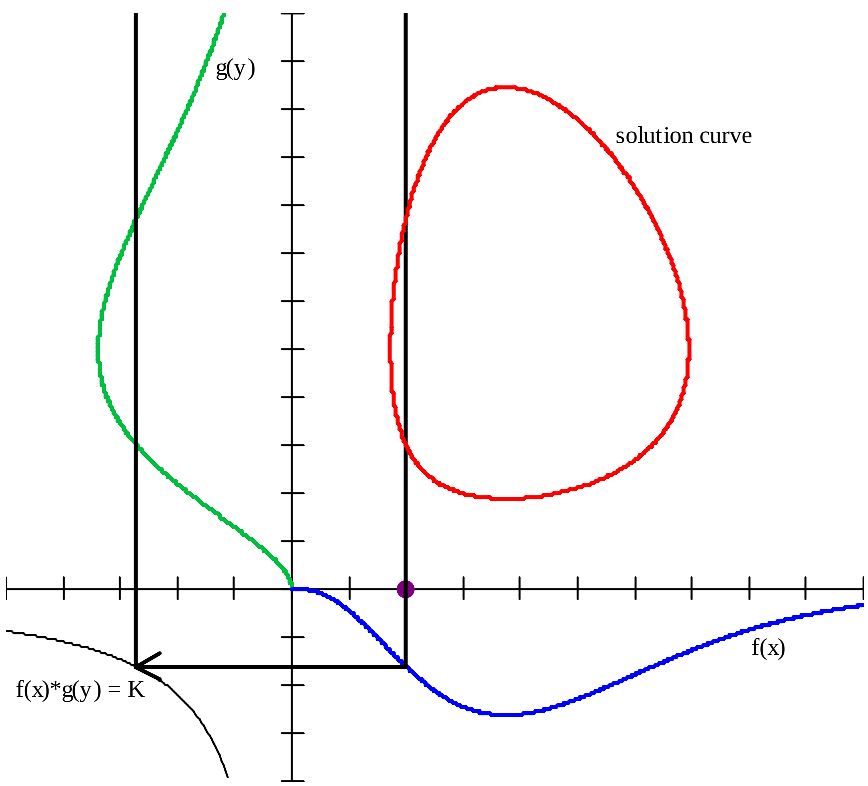
$$\frac{dy}{dx}=\frac{y(-3+0.8x)}{x(2-0.4y)}, \quad y = 3 \text{ when } x = 2$$

Separating and antidifferentiating yields

$$2\ln y-0.4y=-3\ln x+0.8x+C \text{ or } \left(x^3e^{-0.8x}\right)\left(y^2e^{-0.4y}\right)=K \text{ where } K=2^3e^{-1.6}3^2e^{-1.2}=72e^{-2.8}.$$

Let $f(x)=x^3e^{-0.8x}$ and $g(y)=y^2e^{-0.4y}$. The graphs are plotted below with the understanding that the positive axis for $f(x)$ faces down and the positive axis for $g(y)$ faces to the left. The black dot on the *x*-axis represents a possible *x* value on the solution curve. Follow the lines from that point to the blue $f(x)$ curve, to the curve $f(x)g(y)=K$, and then up. The two points on the green $g(y)$ curve where the black line intersect correspond to the two *y* values such that the points (x,y) are on the red solution curve shown in the figure. The black dot can only correspond to an *x*-value on the solution curve as long as the black vertical line intersects the green $g(y)$ graph. The relationship $f(x)g(y)=K$ is essentially like a conservation law that forces the points (x,y) to follow a closed curve in the phase plane.

Note that the maximum of $f(x)$ and $g(y)$ occur at the coordinates of the equilibrium point.



What do solutions look like *near* the equilibrium point at (3.75, 5)? One way to investigate the behavior is to *linearize* the system of differential equations near this point.

Define new variables $v = x - 3.75$ and $w = y - 5.$ Then substitute these into each differential equation and simplify to get

$$\frac{dv}{dt} = \frac{dx}{dt} = 2(v + 3.75) - 0.4(v + 3.75)(w + 5) = -1.5w - 0.4vw$$
$$\frac{dw}{dt} = \frac{dy}{dt} = -3(w + 5) + 0.8(v + 3.75)(w + 5) = 4v + 0.8vw$$

For solutions near the equilibrium point, both *v* and *w* are small, and therefore the product *vw* is very small. So let's ignore this term in each differential equation. This approximation gives the linearized system

$$\frac{dv}{dt} = -1.5w$$
$$\frac{dw}{dt} = 4v$$

Another way to view these two equations is $v'' = -1.5w' = -6v$ and $w'' = 4v' = -6w$. What functions have derivatives that obey such conditions? They are $\sin(\sqrt{6}\,t)$ and $\cos(\sqrt{6}\,t)$, and linear combinations of these two trig function. Solution methods for systems of linear differential equations will show that the general solution to the linearized system given above is

$$v(t) = C_1 \sin(\sqrt{6}\,t) + C_2 \cos(\sqrt{6}\,t)$$
$$w(t) = -\frac{2}{3}\sqrt{6}\left(C_1 \sin(\sqrt{6}\,t) - C_2 \cos(\sqrt{6}\,t)\right)$$

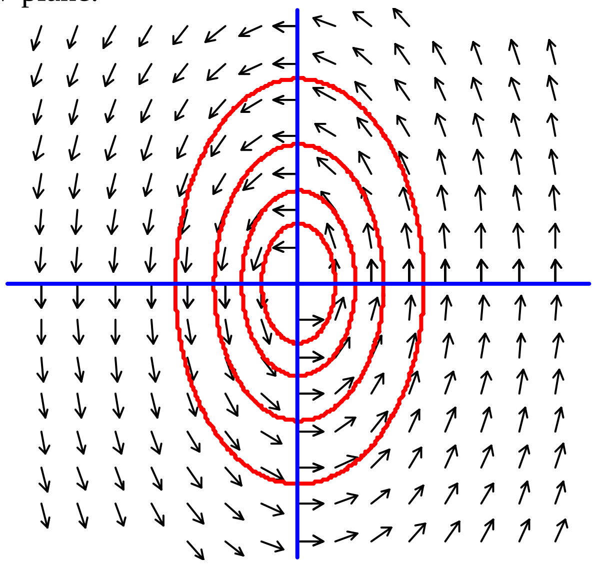
which can be verified by substituting into the equations. Thus for solutions near the equilibrium point in our original system,

$$x(t) \approx C_1 \sin(\sqrt{6}\,t) + C_2 \cos(\sqrt{6}\,t) + 3.75$$
$$y(t) \approx -\frac{2}{3}\sqrt{6}\left(C_1 \sin(\sqrt{6}\,t) - C_2 \cos(\sqrt{6}\,t)\right) + 5$$

The constants C_1 and C_2 are determined by the initial condition. This means that solution curves in the phase plane near the equilibrium point are approximately elliptical, satisfying

$$(x - 3.75)^2 + \frac{3}{8}(y - 5)^2 = {C_1}^2 + {C_2}^2$$

with period $2\pi/\sqrt{6} \approx 2.5651$. The figure below shows four solution curves near the equilibrium point at (3.75, 5). The window has width 0.1 in each direction. The two lines are $x = 3.75$ and $y = 5$, the "axes" in the *vw*-plane.



References:

A.J. Lotka, "Undamped Oscillations Derived from the Law of Mass Action," *Journal of the American Chemical Society*, vol. 42, 1920, pp. 1595—1599.

A.J. Lotka, *Elements of Physical Biology*, Williams & Wilkins, Baltimore, 1925 [Reprinted as *Elements of Mathematical Biology*, Dover Publications, 1956]

V. Volterra, *Lecon sur la Theorie Mathematique de la Lutte pour la Vie*, Gautier-Villars, 1931.

Robert Borrelli and Courtney Coleman, *Differential Equations: A Modeling Perspective*, Wiley, 1998.

"In 1926, Humberto D'Ancona, an Italian biologist, completed a statistical study of the changing populations of various species of fish in the northern reaches of the Adriatic Sea. His estimates of the populations during the years 1910-1923 were based on the numbers of each species sold on the fish markets of the three ports Trieste, Fiume, and Venice. D'Ancona assumed…that the numbers of the various species in the markets reflected the relative abundance of the species in the Adriatic.

As often happens, the data do not provide overwhelming support for any particular theory of changing fish population. D'Ancona observed, however, that the percentages of predator species were generally higher during and immediately after World War I (1914-1918)…Unable to give a reason for the phenomenon, D'Ancona asked his father-in-law, the noted Italian mathematician Vito Volterra (1860-1940), if there was a mathematical model that might cast some light on the matter. Within a few months, Volterra had outlined a series of models for the interactions of two species….A. J. Lotka, an American biologist and, later in life, an actuary, arrived at many of the same conclusions independently of Volterra." [pp287-288]

Competition Model

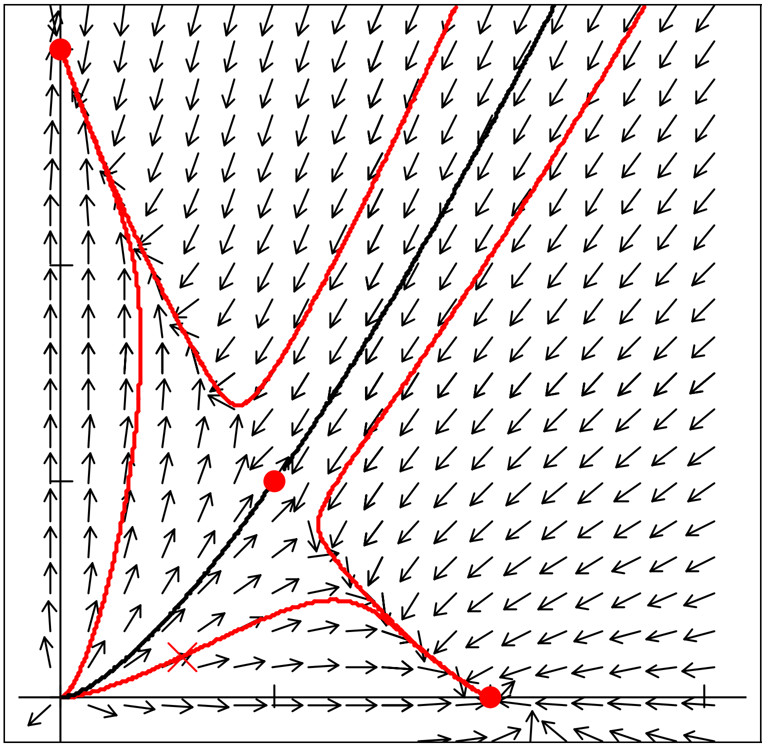
$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy$$

Let $x(t)$ and $y(t)$ be two populations at time *t* that compete for resources.

Assumptions:

3. Each population grows according to a logistic model in the absence of the other population.
4. The number of encounters between the two populations is proportional to the product of their populations. Each such encounter tends to inhibit the growth rate of the populations because of competition.



The three red dots are the equilibrium solutions at (0,3), (2,0), and (1,1). The equilibrium at (1,1) is unstable.

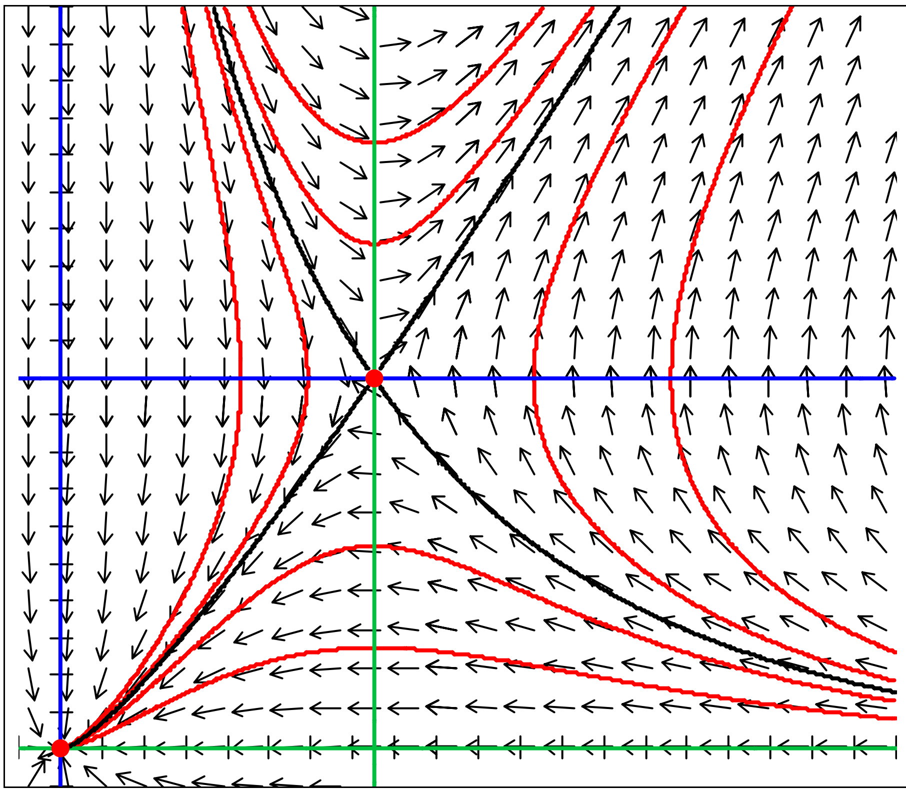
The black curve is a separatrix. This solution curve separates those initial conditions that lead to the *x* population going extinct from those initial conditions that lead to the *y* population going extinct.

Mutualism

$$\frac{dx}{dt} = -x + 0.1xy$$
$$\frac{dy}{dt} = -1.5y + 0.2xy$$

Each population will go extinct in the absence of the other population, but working together will increase the growth of both populations at a rate proportional to the contact between the two populations. [Example: plants and pollinators]

Equilibrium solutions at $(0,0)$ and $(7.5,10)$.



Blue Lines: nullcline for $\frac{dx}{dt} = 0$

Green Lines: nullcline for $\frac{dy}{dt} = 0$

Red Dots: equilibrium points

Red Curves: solution curves

Black Curves: separatices

The separatices are the solution curves in the phase plane that satisfy the implicit equation

$$-\ln y + 0.1y + 1.5\ln x - 0.2x = -\ln 10 + 1.5\ln 7.5 - 0.5$$

obtained by solving $\frac{dy}{dx} = \frac{y(-1.5+0.2x)}{x(-1+0.1y)}$ with the initial point $(7.5,10)$. The separatrices going from upper left to the equilibrium point $(7.5,10)$ and from lower right to the equilibrium point are the solution curves that separate populations that thrive from those that go extinct.

Other Mutualism Models

$$\frac{dx}{dt} = -ax + r_1 xy\left(1 - \frac{x}{K_1}\right)$$
$$\frac{dy}{dt} = -bx + r_2 xy\left(1 - \frac{y}{K_2}\right)$$

The mutualism benefit from the other population becomes a negative effect on the rate of growth if the size of that populations exceeds some value (K_1 or K_2, respectively).

$$\frac{dx}{dt} = r_1 x\left(1 - \frac{x}{K_1 + ay}\right)$$
$$\frac{dy}{dt} = r_2 y\left(1 - \frac{y}{K_2 + bx}\right)$$

Each population obeys a logistic model with the modification that each species has its carrying capacity increased by the presence of the other. Requires $ab < 1$ or the system can grow unboundedly large. The equilibrium point in which both populations are non-zero is

$$\left(\frac{K_1 + aK_2}{1 - ab}, \frac{K_2 + bK_1}{1 - ab}\right).$$

Reference:

Robert May, “Models for Two Interacting Populations”, in *Theoretical Ecology, Principles and Applications*, 2nd Edition, Sinauer Associates, Inc., 1981