Some Thoughts on 2003 AB6

Theorem 1: Suppose g is differentiable on an open interval containing $x = c$. If both $\lim_{x \to c-} g'(x)$ and $\lim_{x \to c+} g'(x)$ exist, then the two limits are equal and the common value is $g'(c)$.

Proof. Let $L_1 = \lim_{x \to c-} g'(x)$ and $L_2 = \lim_{x \to c+} g'(x)$. By the Mean Value Theorem, for every positive h sufficiently small, there exists d_h satisfying $c < d_h < c + h$ such that

$$\frac{g(c+h) - g(c)}{h} = g'(d_h).$$

Then

$$g'(c) = \lim_{h \to 0+} \frac{g(c+h) - g(c)}{h} = \lim_{h \to 0+} g'(d_h) = \lim_{x \to c+} g'(x) = L_2.$$

Similarly, for every positive h sufficiently small, there exists e_h satisfying $c - h < e_h < c$ such that

$$\frac{g(c) - g(c-h)}{h} = g'(e_h).$$

Then

$$g'(c) = \lim_{h \to 0+} \frac{g(c) - g(c-h)}{h} = \lim_{h \to 0+} g'(e_h) = \lim_{x \to c-} g'(x) = L_1.$$

This shows that $L_1 = L_2 = g'(c)$.

Note: The same proof can be modified to show that if g is continuous at $x = c$ and differentiable on both sides of $x = c$, and if $\lim_{x \to c-} g'(x) = \lim_{x \to c+} g'(x) = L$, then g is differentiable at $x = c$ with $g'(c) = L$.

Theorem 2: Suppose p and q are defined on an open interval containing $x = c$ and each are differentiable at $x = c$. Let

$$f(x) = \begin{cases} p(x) & \text{for } a < x \le c \\ q(x) & \text{for } c < x < b. \end{cases}$$

Then f is differentiable at $x = c$ if and only if $p(c) = q(c)$ and $p'(c) = q'(c)$.

Proof : We know that $f'(c)$ exists if and only if $\lim_{x\to c-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c+}\frac{f(x)-f(c)}{x-c}$. We have that

$$\lim_{x\to c-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c-}\frac{p(x)-p(c)}{x-c}=p'(c).$$

Also

$$\lim_{x\to c+}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c+}\frac{q(x)-p(c)}{x-c}=\lim_{x\to c+}\left(\frac{q(x)-q(c)}{x-c}+\frac{q(c)-p(c)}{x-c}\right)=q'(c)$$

if and only if $p(c)=q(c)$. So *f* will be differentiable at $x=c$ if and only if $p(c)=q(c)$ and $p'(c)=q'(c)$.

2003 AB6, part (c) (See question on AP Central for complete statement of problem)

Method 1: We are told that *g* is differentiable at *x* = 3, and so *g* is certainly differentiable on the open interval $(0,5)$.

$$\lim_{x\to 3^-}g'(x)=\lim_{x\to 3^-}\frac{k}{2\sqrt{x+1}}=\frac{k}{4} \quad \text{and} \quad \lim_{x\to 3^+}g'(x)=\lim_{x\to 3^+}m=m$$

So the two limits both exist and by Theorem 1 must be equal. Hence $\frac{k}{4}=m$. Since *g* is continuous at $x=3$, $2k=3m+2$. This gives the two equations to solve for *k* and *m*.

Method 2: Let $p(x)=k\sqrt{x+1}$ and $q(x)=mx+2$. Both are differentiable at $x=3$. If *g* is differentiable at $x=3$, then Theorem 2 implies that $p(3)=q(3)$ and $p'(3)=q'(3)$. This yields the two same two equations as method 1.

Either the note after Theorem 1 or Theorem 2 can be used to show that if we choose $k=\frac{8}{5}$ and $m=\frac{2}{5}$, then we can prove that *g* is differentiable at $x=3$.